Quasi-concave functions and concave functions.

- If $f$ is concave, then it is quasi-concave, so you might start by checking for concavity.
- If $f$ is a monotonic transformation of a concave function, it is quasi-concave. This also means that if a monotonic transformation of $f$ is concave, then $f$ is concave.
- Example: Check whether the $f(x, y) = xy + x^2y^2 + x^3y^3$ defined on $\mathbb{R}^2_+$ is quasiconcave. Note that $f(x) = g(u(x, y))$ where $u(x, y) = xy$ and $g(z) = z + z^2 + z^3$. Since $g' > 0$, $f$ is quasi-concave if and only if $u$ is quasi-concave. But $u(x, y) = e^{v(x,y)}$ where $v(x, y) = \ln x + \ln y$. The function $v$ is easily seen to be concave. So then

$$f(x) = g(u(x, y)) = g(e^{v(x,y)})$$

is a monotone increasing function of a concave function and hence is quasi-concave.
Necessary condition for quasi-concave function.

Let $f$ be a twice continuously differentiable function of $n$ real variables. The bordered Hessian matrix of $f$ looks like this.

$$
H(x) = \begin{bmatrix}
0 & f_1(x) & f_2(x) & \ldots & f_n(x) \\
f_1(x) & f_{11}(x) & f_{12}(x) & \ldots & f_{1n}(x) \\
f_2(x) & f_{21}(x) & f_{22}(x) & \ldots & f_{2n}(x) \\
& \vdots & \vdots & \ddots & \vdots \\
f_n(x) & f_{n1}(x) & f_{n2}(x) & \ldots & f_{nn}(x)
\end{bmatrix}
$$

A necessary condition for $f$ to be a quasi-concave function is that the even-numbered principle minors of the bordered Hessian be non-negative and the odd-numbered principle minors be non-positive.

A sufficient condition for $f$ to be quasi-concave is that the even-numbered principle minors of the bordered Hessian be strictly positive and the odd-numbered principle minors be strictly negative.
Supporting hyperplane theorem

- If $X$ is a convex subset of $\mathbb{R}^n$ and $x_0$ is a point in the boundary of $X$, then there exists a non-zero vector $p \in \mathbb{R}^n$ such that $px \geq px_0$ for all $x \in X$.

- Suppose preferences are convex. Then $X \supseteq (x_0)$ is a convex set. If preferences are monotonic, then $x_0$ is on the boundary of $X$. Then according to the theorem, there is some $p$ such that if $x \succeq x_0$, then $px \geq px_0$. 
Separating hyperplane theorem

- If $X$ and $Y$ are disjoint, non-empty convex subsets of $\mathbb{R}^n$, then there exists a non-zero vector $p \in \mathbb{R}^n$ and a scalar $b$ such that $px \geq b$ for all $x \in X$ and $py \geq b$ for all $x \in Y$. 