Financial Contracts, Bargaining and Security Design[*]

Matan Tsur[†]

Preliminary draft, comments are welcomed.

Abstract

We explore how a firm optimally finances multiple projects when profits depend on subsequent negotiations with buyers, a common situation in procurement projects and vertical industries. A tension arises because the agreements reached with buyers depend on how the proceeds from the sales will be divided with investors. A financial contract determines the division. Should the proceeds from multiple projects be combined and divided jointly or separated and divided independently? Which securities implement the optimal division? In some common conditions the optimal contract takes a simple form. The firm finances each project separately with debt.

1 Introduction

Many transactions require investment and the terms of trade are determined ex-post through negotiations. In vertical industries, input suppliers making specific investments are often locked-in negotiations with down-stream producers. In procurement projects, terms of trade are frequently negotiated (or renegotiated) after investments are made. This paper presents a model to study the interaction of financing and bargaining in these transactions. The main purpose is to characterize the optimal financial contracts that emerge.

Previous literature on the boundary of the firm, the allocation of property and control rights, and incomplete contracts has intensively studied ex-ante investment decisions. The question of how to finance these investments has not received much attention. We consider a generic setting and depart from previous work by allowing the investment to be financed externally. The model captures important trade-offs underlying financing decisions.

In our model, a firm seeks to finance multiple projects, the output of each one will be sold to a specific buyer. Investors will receive some of the proceeds from the sales and the firm will keep the remainder. A financial contract, in the form of securities, determines the division. For example, if projects are financed with debt, investors receive the entire proceeds from the sales until the debt is fully repaid and the additional proceeds beyond this amount will go to the firm. With equity each party receives a fraction of the proceeds. Another security may divide the proceeds differently.

After the contract is signed and investment is made, the firm will negotiate prices with buyers of the outputs. A basic tension arises because the agreements reached with buyers depend on how the proceeds from the sales will be divided with investors. We solve for the contracts that implement the optimal division for the firm. Should the proceeds from different projects be bundled together and divided jointly or separated and divided independently? Which securities should the firm use?

In some common conditions the optimal contract takes a simple form. The firm should finance each project separately with debt. The arrangement is

regularly used in various industries. Large resource extraction or infrastructure projects, for example, are often financed separately. Firms typically set-up independent subsidiaries that can take out specific loans tied to a specific project, the liabilities are separated from the firm's other projects and assets (Esty (2004)). Leveraged buyouts are primarily financed on a deal by deal basis with independent loans (Kaplan and Strömberg (2009)). The following example demonstrates the tensions underlying our results.

> Consider a firm with output worth 200 and a debt of 100 which is repaid from the proceeds of the sales. First, if the output is sold to a single buyer, since the buyer will not trade if the price is above her value and the firm will get nothing if the price is below the debt, the agreement price will fall between 200 and 100. Assuming equal bargaining power, the buyer will pay 150. Without debt in comparison, the buyer will pay half her value, 100.

> Second, if the output is evenly distributed in two markets, the firm will negotiate with two buyers, each with value 100. Can it still extract a surplus of 150? The answer is no. Suppose transactions occur in sequence and the first buyer pays 75, since a debt of 25 remains, the second buyer only pays 62.5 (the midpoint between 25 and 100) and the proceeds are less than 150. Basically, the more the first buyer pays, the lower is the remaining debt and the less the second buyer pays.

That is, a firm with debt can extract a large share of the surplus from a single buyer, but when there are multiple buyers, externalities arise and the advantage is attenuated.

There are a variety of ways to finance multiple projects. The firm can take-out a single loan which is repaid from the joint proceeds of the sales. Another option is to finance projects separately with multiple loans. When each loan tied to a specific project, transactions are independent because an agreement with one buyer will not effect the the loans. The firm can also design other securities. A security determines the amount payed to investors as a function

3

of the the proceeds from one or several projects. The payment to investors is positive and increases with the proceeds. The analysis characterizes the bargaining outcomes for the different securities.

We begin with the case of a single buyer. Since some of the proceeds from the sale go to investors, the shape of the security determines the slope of the feasible payoff frontier that the firm and buyer can achieve. We let Rubinstein's (1982) sequential bargaining game pin down the outcomes and find that debt is the optimal security in this case. That is, the bargaining game with debt has a subgame perfect equilibrium that achieves the firm's maximal payoff.

Intuitively, it is not the firm's overall share of the proceeds that impacts the bargaining outcomes, but rather how this share changes. For example, with a linear security the firm will get a constant share of the proceeds, whereas a security that increases the firm's share with the proceeds creates stronger bargaining incentives, because the firm will keep a larger share of a higher price. With debt the proceeds are divided sequentially with investors payed first, so the firm's share increases steeply.

However, a firm with multiple projects will negotiate with multiple buyers. A security that is a function of the joint proceeds from the sales creates a link between the transactions, the question is whether the firm can design the security to benefit from externalities. For simplicity, we assume that the bargaining outcomes are bilaterally stable, each price is an outcome of bilateral bargaining taking the other prices as given.[1]

While a complex structure of entangled agreements with buyers may arise, the effect of the externalities boils down to a basic property of the security: concavity. When securities are concave functions of the proceeds, the firm should keep transactions independent and finance projects separately with debt. However, there exists a security which is a non-concave function of the joint proceeds that implements a better outcome. This security back-loads the payments to investors, similar to a call option.

To see why concavity matters, notice that the agreement reached with a certain buyer depends on the *additional* payment to investors. When the security

---

[1]We borrow the term from Lensberg (1988), the property is due to Harsanyi (1963).

is a concave function of the joint proceeds, the higher the proceeds in the other transactions, the lower the additional payment to investors from this transaction and the lower the price. The externalities are negative and the firm is better-off eliminating them by financing each project independently. [2]

One caveat of bargaining theory is that outcomes can be sensitive to the bargaining model which is used. We also consider other bargaining models – for bilateral problems, the canonical solutions of Nash (1950) and Kalai and Smorodinsky (1975); for multilateral problems, we characterize the equilibrium (with refinement) of a strategic bargaining game – and find that the results are robust. Finally, we study some welfare implications of our model in two common settings. In markets where hold-up problems reduce the potential gains from trade, we find that debt-financing may strengthen the incentives to make ex-ante investments. On the other hand, debt may also create inefficient distortion in ex-post allocations, influencing how much an exporter will trade with local distributors for example.

2   Related Literature

This paper relates to several strands of the literature. There is an extensive literature on bargaining theory and its applications. In Stole and Zwiebel (1996a,b), for example, a firm's internal organization and technological decisions are made to influence wage negotiations with each of its workers. In a similar vein, we extend this literature to study financing decisions in some common situations where a different problem arises, how to optimally divide proceeds that accumulate from bargaining.

The security design literature has focused on different frictions, mainly due to incomplete information such as moral hazard, adverse selection and costly state verification.[3] The advantages and costs of debt contracts are well known.

---

[2]The basic securities, debt and equity, are concave and securities which are convex combinations of them are concave as well. To implement a non-concave payment schedule, firms must use more sophisticated securities, such as option contracts, that are not always available.

[3]Some classic papers include Jensen and Meckling (1979), Townsend (1979), Myers and

In our model there is complete information, no risk or uncertainty and neither agency nor bankruptcy costs. We find that debt is optimal when negotiations are bilateral, but the advantage is attenuated when negotiations are multilateral.

In terms of the underlying tensions, the present model is more closely related to the literature on the inter-linkage between debt and transactions in output markets. The seminal paper of Brander and Lewis (1986) shows that debt creates a strategic advantage for a firm competing with rivals. The following literature mainly focuses on competitive output markets (e.g. Bolton and Scharfstein (1990), Chevalier (1995)). Our analysis pertains to non-competitive markets where terms of trade are typically determined via negotiations. Interestingly, we find that when there are sufficiently many buyers, the advantage of debt may not be preserved in our setting.

The basic interaction between financing decisions and ex-post negotiations is common in the corporate finance literature, the negotiations are typically between borrowers and creditors when liquidation and refinancing decisions are made (e.g. Rajan (1992), Hart and Moore (1997)). More closely related are a number of papers on the effect of debt on collective bargaining agreements. The main point is that the firm can decrease its cash flow by taking on more debt, and as a result, less surplus can be extracted by the union (e.g. Bronars and Deere (1991), Matsa (2010)). Our model considers a different setting where the firm's investment generates surplus to outside parties (the buyers) and the financial contract divides the proceeds from the sales, it has no affect on the buyers' surplus. Different mechanisms are at work and this distinction matters: debt may have very little impact (if at all) in our setting.

Part of our contribution is technical. To solve for the optimal securities, the outcomes of a large class of bargaining problems are characterized. For clarity, we use a reduced-form model based on bilateral stability and to show robustness, we also characterize the equilibrium of a strategic bargaining game. There are powerful theories concerning these games, several papers show that

Majluf (1984), Gale and Hellwig (1985), Innes (1990), DeMarzo (2005) and some more recent ones, Axelson (2007), Antic (2014). DeMarzo et al. (2005) study security-bid auctions.

the static and dynamic solutions are close in various bargaining situations (Krishna and Serrano (1996), Hart and Mas-Colell (1996), Collard-Wexler et al. (2014)), but previous analysis does not apply to the bargaining problems in our model. The reason is that the structure of externalities in our model differs and the feasible payoff sets need not be convex. Our analysis may prove useful in other environments.

3 Model

A single firm has N projects that require investment, the investment in project j is denoted by ξ_j. The output of each project will be sold to a unique buyer. Buyer j values the output of project j by v_j and does not value the output of another project. Projects are profitable $0 < \xi_j < v_j$ and the total surplus and investment are denoted by $V = \sum_{j=1}^{N} v_j$ and $I = \sum_{j=1}^{N} \xi_j$. In the first stage, the firm decides how to finance the projects and in the second stage, the firm negotiates prices with the buyers. The two ingredients, financing and bargaining, are described below.

The firm can finance projects through a competitive market of investors who will receive some of the proceeds from the sales in return. A *security* $S : \mathbb{R} \to \mathbb{R}^+$ determines the payment to investors as a function of the proceeds from the sales.[4] We focus on two types of financial contracts:

1. We say that projects are financed *jointly* if the contract consists of a single security S which is a function of the aggregate revenues.

2. We say that projects are financed *separately* if the contract consists of N securities $S_1, \ldots, S_N$ and for all j, security S_j is a function only of the proceeds from project j.

Two common restrictions are imposed. The payment to investors is positive $S \geq 0$ and weakly increases with the revenue $S' \geq 0$. The first restriction rules

---

[4]Note that the security specifies the total amount payed to investors, their number does not matter.

out transfers from the investors to the firm. Such transfers are rarely observed because they expose investors to various risks. For example, firms will seek to finance unprofitable projects just to receive a positive transfer, increasing screening and monitoring costs for investors. The second restriction, that $S' \geq 0$, is usually justified on the premise that otherwise the firm will have an incentive to artificially inflate the proceeds and pay the investors less.[5] While both restrictions naturally arise in a more general setting, we exogenously impose them for clarity.

After the financial contract is signed and investment is made, the firm negotiates prices with buyers of the outputs. An agreement with buyer j refers to the price payed by the buyer. The proceeds from the sales will be divided according to the financial contract. For example, following the agreements with buyers $1, \ldots, K$ over prices $x_1, \ldots, x_K$, the firm will get $X_K - S(X_K)$ where $X_K = \sum_{i=1}^{K} x_i$ with a joint contract and $\sum_{i=1}^{K} x_i - S_i(x_i)$ with a separate contract. Buyer $j \leq K$ will get $v_j - x_j$. If no agreement is reached with some buyers, those buyers get nothing and the proceeds from those projects are zero. No generality will be lost if we set $S(0) = 0$ and $S_j(0) = 0$.

For a joint and a separate contract, let $G(S)$ and $G(S_1, \ldots, S_N)$ denote the set of feasible payoffs that the firm and buyers can achieve from the possible agreements. *A bargaining solution $\mathcal{F}$ selects a set of agreements, denoted by $\mathcal{F}G(S)$ and $\mathcal{F}G(S_1, \ldots, S_N)$, from a feasible payoff set. Figure 1 that depicts the utility frontier when there is a single buyer. The basic tension is that the division rule, determined by the security, changes the slope of the utility frontier.

The firm designs the financial contract taking into account the effect on subsequent negotiations and that investors at least break even. Given a bargaining solution $\mathcal{F}$, we say that a contract is *optimal* if the corresponding bargaining game has an outcome such that 1) the investors at least break even and 2) there does exist another contract where the corresponding bargaining game has an outcome that makes the firm better-off and the outside investors at least break even.

---

[5]With a decreasing security, for example, the firm profits from selling output to itself because $X - S(X + \epsilon) > X - S(X)$.

Figure 1: Utility Frontier
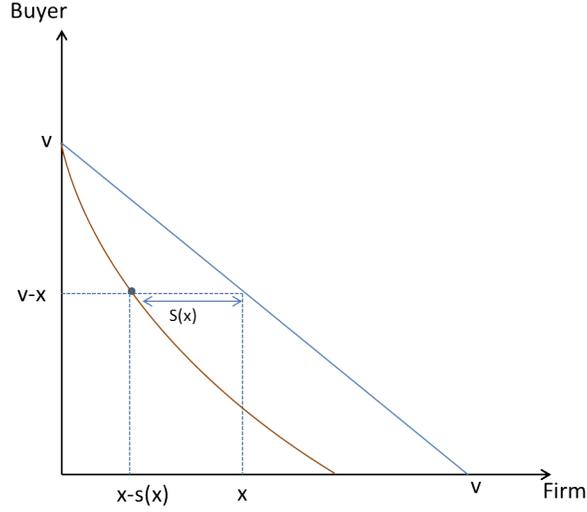
Thus, with a joint contract, the optimal security solves

$$\max_{S} \quad X - S\left(X\right)$$

$$\text{s.t.} \quad \left(x_1, \ldots, x_N\right) \in \mathcal{FG}\left(S\right) \text{ and } X = \sum_{i=1}^{N} x_i$$

$$S(X) \geq I \text{ and } S, S' \geq 0$$

and with a separate contract, the optimal securities solve[6]

$$\max_{S_1, \ldots S_N} \quad \sum_{i=1}^{N} x_i - S_i\left(x_i\right)$$

$$\text{s.t.} \quad \left(x_1, \ldots, x_N\right) \in \mathcal{FG}\left(S_1, \ldots, S_N\right)$$

$$\forall i: \ S_i(x_i) \geq \xi_i \ \text{ and } S_i, S_i' \geq 0$$

In words, this is a simple division problem. The proceeds from multiple

---

[6]Our analysis and results do not change if we allow for cross-subsidization between projects and require $\sum_{i=1}^{N} S_i(x_i) \geq I$ instead of $S_i(x_i) \geq \xi_i$.

9

projects can either be combined and divided jointly or separated and divided independently. In each case, we are looking for the division rule that maximizes the firm's payoff under the restrictions that the payment to investors is sufficiently large and increases with the proceeds.

There is complete information in the model, the costs of projects, the values of buyers and the financial contracts are publicly observed. The bulk of the analysis focuses on specific bargaining solutions. For bilateral negotiations, we solve for the subgame perfect equilibrium agreements in the sequential bargaining game of Rubinstein (1982), and for multilateral negotiations, we use a reduced form model based on bilateral stability. Formal descriptions are given in the next sections. There are various bargaining models and one caveat of bargaining theory is that the outcomes in $\mathcal{F}G(S)$ may be sensitive to the modeling choice. For robustness, we will also extend the analysis to other bargaining solutions and show that our results are preserved.

Remark. The two types of contracts, joint and separate, capture a common organizational decision of whether to incorporate multiple investments within the firm or undertake them separately, through independent subsidiaries for example. If multiple projects are financed within the firm, then it may not be possible to provide lenders with independent claims for specific projects. In order to do so, firms often set-up subsidiaries. Theoretically, more general contracts can also be considered. For one, the firm should be able to finance only a subset of projects separately and the others jointly. No generality is lost in focusing on the two extreme partitions above. More general contracts, however, could be written if securities would depend on the vector of revenues instead of the sum, i.e. $S(x_1, \ldots, x_N)$. The analysis and results are easily extended to this case in the Appendix.[7]

---

[7]Note that if multiple projects are financed within the firm and it is possible to shuffle the proceeds from one project to another project, then the payment to investors would only depend on the sum of the proceeds.

4 Optimal contract for a single project

A firm with a single project will negotiates with a single buyer. We let the sequential bargaining game of Rubinstein (1982) pin down the outcomes. The firm and buyer alternate offers in the usual way and following a rejected offer, the game continues with probability $p < 1$ and breakdown occurs with probability $1 - p$. The game ends when either an offer is accepted or a breakdown occurs. The payoffs are given by $G(S)$ and the bargaining outcomes $\mathcal{F}G(S)$ are the subgame perfect equilibrium agreements when the probability of a breakdown vanishes. The solution is well defined and always exists (see appendix).

Consider the bargaining game with debt $S(x) = \min[x, D]$ for example, the firm will get the additional proceeds after the debt is fully repaid. There is a subgame perfect equilibrium where the buyer offers the price[8] $x_B = D + \frac{p}{1+p}(v - D)$ and the firm $x_F = D + \frac{1}{1+p}(v - D)$, the offers converge to the price $x = \frac{v+D}{2}$ as $p \to 1$. Intuitively, the buyer will not trade if the price is above her value and the firm will get nothing if the price is below the debt, since $p \to 1$ there is equal bargaining power, so the agreement falls exactly in the middle between D and v. When $D = I$, investors break even and the firm and buyer split the social surplus, each gets $\frac{1}{2}(v - I)$.

In the bargaining game with equity $S(x) = \alpha x$ and $\alpha < 1$, the firm will get a fraction $1 - \alpha$ of the proceeds. There is a subgame perfect equilibrium with prices $x_B = \frac{p}{1+p}v$ and $x_F = \frac{1}{1+p}v$, the offers converge to the price $x = \frac{v}{2}$ as $p \to 1$ and the firm will get $(1 - \alpha)\frac{v}{2}$. Notice that increasing the firm's share will increase its payout, but will not change the equilibrium price, the reason is that the firm's share changes by the same amount for all prices. Investors break even when $\alpha\frac{v}{2} \geq I$ and the firm will get at most $\frac{v}{2} - I$.

Thus, the firm can extract a larger share of the surplus with debt than with equity. Our first result extends the advantage of debt to all securities.

Proposition 1. *Financing a single project with debt is optimal.*

---

[8]The strategies are stationary: the buyer and firm always offer prices x_B and x_F and each player only accepts an offer when the payoff is no less than the continuation payoff assuming her offer is accepted in the next period.

The proof bounds the firm's equilibria payoffs and then shows that debt achieves this bound. Lemma 1 does most of the work.

Lemma 1. *Given a security S, if m_F and m_B are SPE payoffs of the firm and the buyer, then $p \times m_F \leq m_B$.*

Proof. Fix a security S. It is without the loss of generality to assume that $x - S(x)$ weakly increases. Let $\overline{x}_F$ and $\overline{x}_B$ be the maximal SPE prices in a subgame beginning with an offer made by the firm and the buyer respectively. The maximal prices are achieved in an equilibrium without delay. An equilibrium price offer x by the buyer satisfies $x - S(x) \leq p(\overline{x}_F - S(\overline{x}_F))$ and a price accepted by the buyer satisfies $p(v - \overline{x}_B) \leq v - x$. Since $S, S' \geq 0$ and $\overline{x}_B \leq \overline{x}_F$ we have that

$$\overline{x}_B \leq p\overline{x}_F + (1 - p)S(\overline{x}_B) \text{ and } \overline{x}_F \leq (1 - p)v + p\overline{x}_B$$

These imply

$$p(\overline{x}_F - S(\overline{x}_F)) \leq v - \overline{x}_F \text{ and } \overline{x}_B - S(\overline{x}_B) \leq p(v - \overline{x}_B)$$

When player i proposes, the firm's equilibrium payoff is no more than $\overline{x}_i - S(\overline{x}_i)$ and the buyer's no less than $v - \overline{x}_i$. $\qquad\square$

Thus, as $p \to 1$, the firm's payoff will not exceed the buyer's and the rest of the proof immediately follows. Investors get at least I, which leaves no more than $V - I$ and since the firm will not get more than the buyer, $\frac{1}{2}(V - I)$ is the firm's maximal payoff.[9] Figure 2 illustrates the argument, the equilibrium payoffs lay above the 45° line and the agreement must fall in the shaded region for investors to break-even. Finally, as demonstrated above, the bargaining game with debt has a subgame perfect equilibrium that achieves the firm's maximal payoff while investors break even.

Proposition 1 hinges on a basic point. It is not the firm's overall share of the price that impacts the bargaining outcomes, but rather how this share

---

[9]The argument trivially extends to any $p < 1$. We take $p \to 1$ to eliminate a first move advantage.

Figure 2: Equilibria Payoffs

....condense figures
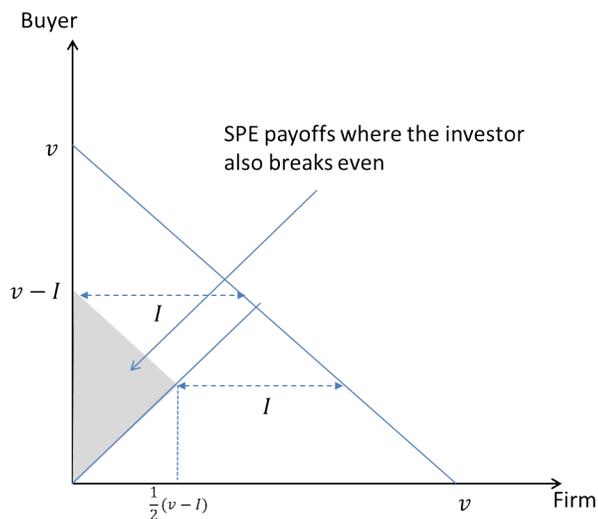
changes. For example, with a linear security, the firm gets a constant share of the price; with a concave security, the firm's share increases with the price ($\frac{x-S(x)}{x} = 1 - \frac{S(x)}{x}$); and with a convex security, the share decreases with the price. It is intuitive that a concave security implements higher prices because a firm that keeps a larger share of a higher a price (the share increases) has stronger bargaining incentives. Moreover, the security that increases the firm's share in the steepest manner is debt.

The result is robust to the specification of the bargaining model. If we let the canonical solutions of Nash (1950) or Kalai and Smorodinsky (1975) select the outcomes in $\mathcal{FG}(S)$, debt remains optimal.[10] These solutions provide a useful geometric interpretation which is given in the Appendix.

Remark 1. If a security strictly increases, the inequality in Lemma 1 is strict and the firm will get less than the buyer. Thus, while the optimal security

---

[10]More precisely, we consider the natural extension of these solutions to problems with non-convex utility sets (see Zhou (1997) and Conley and Wilkie (1991) for details).

is not unique, all optimal securities resemble debt in that the payment to investors is constant over some region.

Remark 2. The canonical axiomatic and strategic bargaining models are closely related when negotiations are bilateral. Under general conditions, the sequential bargaining game of Rubinstein has a subgame perfect equilibrium that is close to (and converges to) the Nash solution, which selects the agreements that maximize the product of the players utilities over the feasible payoff set. But the feasible payoff set $G(S)$ in our model need not be convex and the sequential bargaining game may have multiple equilibria, some of which need not converge to the Nash solution (Herrero (1989)).[11]

In the bargaining game with debt $S(x) = \min[x, D]$, the agreement $x = \frac{v+D}{2}$ maximizes the Nash product uniquely and is supported by the strategic model. But the strategic game also has a class of degenerate equilibria with low prices $x < D$. These equilibria arise because the firm gets nothing from agreements that fall short of the debt and is therefore indifferent over the range of low prices. However, degenerate outcomes are unstable and will not survive simple refinements, such as trembling hand. Moreover, if degenerate outcomes are ruled out, either with refinements or with an indifference assumption,[12] then the strategic bargaining game with debt has a unique outcome that converges (with p) to the price $x = \frac{v+D}{2}$. Under these assumptions, debt implements the firm's maximal payoff uniquely.

Remark 3. A technical method to convexify the payoff set is to allow for randomization. But this is highly unrealistic in our settings and completely changes the trade-offs of the model. With randomization, for example, bargaining with debt is equivalent to bargaining with equity.

Remark 4. The two restrictions on the securities, increasing and positive,

---

[11]For example, consider the security $S(x) = \min[\lambda x, D]$. If $\lambda < 1$ and $\lambda\frac{v}{2} < D < v$, then both $x_1 = \frac{v}{2}$ and $x_2 = \frac{v+D}{2}$ can be supported as an SPE of the corresponding bargaining game, but the former does not maximize the Nash product for λ large.

[12]It is reasonable to assume that the firm would rather wait than reach an agreement when it gains nothing.

are important for the result. For example, consider the security

$$
S(x) = \begin{cases} x & \text{for } x < D \\ I & \text{for } D \leq x \end{cases}
$$

That is, the firm will get nothing if outcomes are low and a large payout when outcomes are high. When D is sufficiently high, $v - \epsilon < D \leq v$ and $\epsilon < \frac{v-I}{1+p}$, the corresponding bargaining game has an SPE with stationary strategies in which the buyer and firm offer the prices $x_B = D$ and $x_F = (1-p)v + px_B$. Notice that S decreases (because $I < v - \epsilon < D$) and the firm can get more than the buyer in this equilibrium. Moreover, the firm can somewhat trivially extract the entire social surplus by setting $D = v$. However, when the payment to investors is positive and increases with the proceeds, a basic tension arises – to extract more surplus from the buyer, the firm must also increase the payment to investors – and the design problem is non-trivial.

5 Optimal Contract for Multiple Projects

The previous section finds that debt implements the optimal division rule when there is a single transaction. When the proceeds accumulate from multiple transactions, externalities may arise. This section examines their effect and solves for the optimal contract.

We begin with a few definitions. When the payment to investors is a function of the joint proceeds from the sales, the gains from trade in one transaction may depend on the outcomes in the other transactions. Let $B_j(S, Y)$ denote the set of feasible payoffs that the firm and buyer j can achieve if the security is S and the joint proceeds in the other transactions are Y. Following an agreement over price x, the joint proceeds are $Y + x$ and the firm will get $Y + x - S(Y + x)$. If an agreement is not reached with this buyer, the firm will get $Y - S(Y)$. The buyer's payoff is standard. As in the previous section, let $\mathcal{F}B_j(S, Y)$ denote the subgame perfect equilibrium agreements in the sequential bilateral bargaining game when the feasible payoff set is $B_j(S, Y)$ and the probability

of a breakdown vanishes.

Definition. We say that prices $x_1, \ldots, x_N$ are *bilaterally stable* if for all j: $x_j \in \mathcal{F}B_j(S, X_{-j})$ where $X_{-j} \equiv \sum_{i \neq j}^{N} x_i$.

That is, each price is the outcome of bilateral negotiations given the other prices. The concept hinges on the intuitive idea that each player can either trade at a given price or wait and negotiate the price later, after the other deals have been settled. If an outcome $x_1, \ldots, x_N$ is bilaterally stable, no player has an incentive to wait and negotiate after the others, and if an outcome is not bilaterally stable, at least one player would rather wait than trade at that price.[13] A bilaterally stable outcome always exists but need not be unique (see proof in the appendix).

Let $\mathcal{F}G(S)$ be the set of bilaterally stable agreements. In this subsection, we do not not explicitly model the process that governs the negotiations, but rather assume that this process delivers a bilaterally stable outcomes. The next subsection extends the analysis to a dynamic bargaining game. The reduced-form approach has the usual advantages, bilaterally stable outcomes are computationally tractable and clearly capture the main economic ideas. The next example demonstrates.

Example 1: Projects are financed jointly with debt $S(X) = \min[X, D]$.

To characterize the bilaterally stable outcomes, notice that the agreement reached with a certain buyer depends on the gains from trade relative to the status quo if an agreement is not reached. If the proceeds in the other transaction are Y and the remaining buyer pays x, then either $Y \geq D$ and the firm will gain exactly what the buyer pays because

$$\underbrace{x + Y - D}_{\text{Agreement}} - \underbrace{(Y - D)}_{\text{Status quo}} = x$$

---

[13]Note that if $\overline{x}, \underline{x}$ are the maximal and minimal SPE prices of the bilateral bargaining game, then a price in between, $\underline{x} \leq x \leq \overline{x}$, is also an SPE of this game. Therefore, if x_j is not an equilibrium price of the continuation game, then either $x_j < \underline{x}$ (and the firm prefers to wait) or $x_j > \overline{x}$ (and the buyer prefers to wait).

or $Y < D$ and the firm gains $x + Y - D$ if $x + Y \geq D$ and nothing if $x + Y < D$. Thus, the bilateral bargaining game with feasible payoffs $B_j(S, Y)$ is equivalent to the bargaining game with a debt of $D' = \max[D - Y, 0]$ and status quo payoffs at the origin. The maximal price in $\mathcal{F}B_j(S, Y)$ is therefore $\frac{1}{2}(v_j + D')$ and prices $x_1, \ldots, x_N$ which satisfy

$$\text{For all } i: \ x_i = \frac{1}{2}(v_i + D_{-i}) \text{ where } D_{-i} = \max[0, D - \sum_{j \neq i}^{N} x_j]$$

are bilaterally stable. The solution is particularly simple when values are symmetric, $v_1 = \ldots = v_N = v$, either

1. $(N - 1)\frac{v}{2} \geq D$, the prices $(\frac{v}{2}, \ldots, \frac{v}{2})$ are bilaterally stable: the revenue in $N - 1$ transactions covers the debt, so a remaining buyer pays $\frac{v}{2}$; or

2. $(N - 1)\frac{v}{2} < D$, the prices $(\frac{v}{2}, \ldots, \frac{v}{2})$ are not bilaterally stable (because a remaining buyer will pay a higher price), but the prices $\left(\frac{v+D}{N+1}, \ldots, \frac{v+D}{N+1} \right)$ are bilaterally stable.

Moreover, it is not hard to verify that these are the maximal bilaterally stable outcomes in the respective cases.[14] Therefore, when projects are financed jointly with debt, investors break even if $D \geq I$ and the firm will get at most

$$\pi = \begin{cases} \frac{V}{2} - I & \text{if } (N - 1)\frac{v}{2} \geq I \\ \frac{1}{N+1}(V - I) & \text{if } (N - 1)\frac{v}{2} < I \end{cases} \tag{1}$$

It is intuitive that the agreement reached with a certain buyer should depend both on the debt level and on how it is expected to change. For example, if the proceeds in the other transactions are sufficiently high, the debt will be repaid regardless of the agreement with this buyer and should not affect it. In the first case, the same logic applies to each

---

[14]There is a subtle point here, it can easily be verified that the firm does not prefer to burn some surplus and negotiate only with a subset $m < N$ of buyers.

transaction and debt affects none of them. In the second case, the effect of debt is preserved, but attenuated. The more buyers there are, the smaller the firm's share of the social surplus. The firm will get much less than half the social surplus, $\frac{1}{2}(V - I)$.

The externalities are the driving force here. When each project is financed with a separate loan, in contrast, the transactions are independent because an agreement with one buyer will not affect the other loans. Our main result finds that this simple arrangement is optimal within a large class of contracts.

Proposition 2. *When securities are concave functions, financing projects separately with debt is optimal.*

Lemma 3 does most of the work, it shows that the firm will get no more than the sum of the buyers' payoffs.

Lemma 3. *If S is concave and prices $x_1, ..., x_N$ are bilaterally stable, then $X - S(X) \leq V - X$ where $X \equiv \sum_{i=1}^{N} x_i$.*

Proof. When the firm trades with a particular buyer at price x and the proceeds in the other transaction are Y, the firm gains

$$\underbrace{Y + x - S(Y + x)}_{\text{Agreement}} - \underbrace{(Y - S(Y))}_{\text{Status quo}} = x - S(Y + x) + S(Y) \qquad (2)$$

from this trade. First, since prices $x_1, \ldots, x_N$ are bilaterally stable, Lemma 1 implies that the firm's gain from each trade does not exceed the buyer's gain,

$$\forall i: \ x_i - (S(X_{-i} + x_i) - S(X_{-i})) \leq v_i - x_i \qquad (3)$$

To see it, note that $x_i \in \mathcal{F}B_i(S, X_{-i})$ and we can transform the bilateral bargaining game with payoffs $B_i(S, X_{-i})$ by subtracting the amount $X_{-i} - S(X_{-i})$ from the firm's payoff. In the resulting bargaining game, the feasible payoffs are $B_i(\hat{S}_i, 0)$ where $\hat{S}_i(y) = S(X_{-i} + y) - S(X_{-i})$ and the status quo payoffs are at the origin. An SPE in one game is also an SPE in the other, $\mathcal{F}B_i(S, X_{-i}) = \mathcal{F}B_i(\hat{S}_i, 0)$, and Lemma 1 implies $x_i - \hat{S}_i(x_i) \leq v_i - x_i$.

18

Second, since S is concave, for all i, $S(X) - S(X_{-i}) \leq \frac{x_i}{X}[S(X) - S(0)]$ and in sum,

$$\sum_{i=1}^{N} S(X) - S(X_{-i}) \quad \leq \quad S(X) - S(0)$$

Therefore, the firm's profit is no more than the sum of the gains from each trade,

$$X - S(X) \leq X - \sum_{i=1}^{N} S(X) - S(X_{-i}) \tag{4}$$

and from (3) and (4) we have that

$$X - S(X) \leq V - X$$

$\square$

Recall that for bilateral bargaining, we showed that the firm will not get more than the buyer for any security (Lemma 1), Lemma 3 extends this result to the multilateral cases, but requires concavity. The rest of the proof immediately follows. As long as the investors get at least I, no more than $V - I$ is left over and since the firm's payoff does not exceed the total payoff of the buyers, $\frac{1}{2}(V - I)$ is an upper bound.

The main point here is that in the negotiations with a certain buyer, gains from trade are assessed relative to the status quo (Equation 2). If a trade occurs at price x and the proceeds in the other transactions are Y, the firm's gain will not exceed the buyer's (Equation 3),

$$x - S(Y + x) + S(Y) \leq v - x \quad \Longrightarrow \quad x \quad \leq \quad \frac{1}{2}[v + S(Y + x) - S(Y)]$$

Concavity matters because the higher the revenue in the other transactions Y, the lower the *additional* payment to investors and the lower the maximal price. The externalities are negative. The proof may appear deceptively simple, but this is a powerful result. The next example demonstrates.

Example 3: There are two identical projects, each one requires an investment of $\xi = 20$ and generates a surplus of $v = 100$.

- If a debt of $D = I = 40$ is tied to the joint proceeds, each buyer will pay 50 and the firm will get 60. Suppose that projects are financed jointly with security

$$S_1(X) = \min[\frac{X}{2}, D] \text{ and } D = 40$$

  This security allocates only half of the revenue towards covering the debt, the rest goes to the firm. In the corresponding bargaining game, each buyer paying 50 is no longer a bilaterally stable outcome: if one buyer pays 50, since only half goes towards the debt, a debt of $D' = 40 - \frac{50}{2} = 15$ remains and the other buyer will pay more than 50 in the continuation game. The prices $x_1 = x_2 = 56$ are bilaterally stable: if one buyer pays 56, a debt of $D' = 40 - \frac{56}{2} = 12$ remains and the other buyer will also pay 56. The firm will get 72 and, importantly, the outside investors still break even.

  Basically, since the debt is repaid from the joint proceeds of the sales, buyers have incentives to wait and negotiate later, when the debt is partially or completely payed out. Security $S(X) = \min[\lambda X, D]$ allocates only a fraction $\lambda < 1$ of the proceeds towards the debt and since the debt decrease at a slower rate, the less attractive it is for buyers to wait. In this example $\lambda = \frac{1}{2}$ and the firm's payoff is 72, more than what the firm would get with debt, 60, but less than half the social surplus, $\frac{1}{2}(V - I) = 80$. Indeed, the firm can get more than 72 by choosing a lower fraction $\lambda < \frac{1}{2}$, but the fraction can not be too low because investors will not break-even. This security is concave and the firm can not get more than 80.

- The firm can get more than half the social surplus with a non-concave security. Suppose that the projects from previous example are financed

jointly with security

$$S_2(X) = \begin{cases} 0 & \text{if } X < 70 \\ \min[X - 70, 40]) & \text{if } X \geq 70 \end{cases}$$

This security back-loads the payments to investors: the proceeds between 0 and 70 go to the firm, the proceeds between 70 and 110 go to investors and the firm receives the additional proceeds. In the bargaining game with this security, each buyer paying 70 is bilaterally stable. The point is that a buyer who waits to negotiate after the other will face a firm that has a debt of 40 and will therefore pay 70, even though the cost of the project is 20. Intuitively, by financing projects jointly and back-loading the payments to investors, the firm can use the joint costs of the projects as bargaining leverage over each buyer.

However, the bargaining game with this security has other bilaterally stable outcomes, the prices $(50, 20)$ for example.[15] That is, there are many bilaterally stable outcomes in $\mathcal{FG}(S_2)$, in one outcome the firm will get more than half the social surplus and investors break even, but in other outcomes the firm and investors get much less.

Financing projects separately with debt is robust. If degenerate outcomes are ruled out (see Remark), then this arrangement implements a bargaining game with a unique outcomes.

Securities S_1 and S_2 are depicted in figure 3.

To sum up, securities may generate a complex structure of exteranlities and a host of entangled agreements may arise. Our main result finds that as

---

[15]To see it, consider the bargaining game between the firm and Buyer 1 after Buyer 2 payed 20. The following stationary strategies are a subgame perfect equilibrium: the firm always offers $x_F = 50$, Buyer 1 always offers $x_B = 50p$ and each party only accepts offers if it gets no less than the continuation value (assuming their offer is accepted in the next period). Note that the buyer strictly prefers to accept the firm's offer than to wait, but the firm will not profit from offering a slightly higher price.

Likewise, in the continuation game with payoffs $B_2(S_2, 50)$, the following stationary strategies are a subgame perfect equilibrium: the firm always offers $x_F = 20$, Buyer 1 always offers $x_B = 20p$ and each party only accepts offers if it gets no less than the continuation value (assuming their offer is accepted in the next period).

Figure 3: Optimal Securities
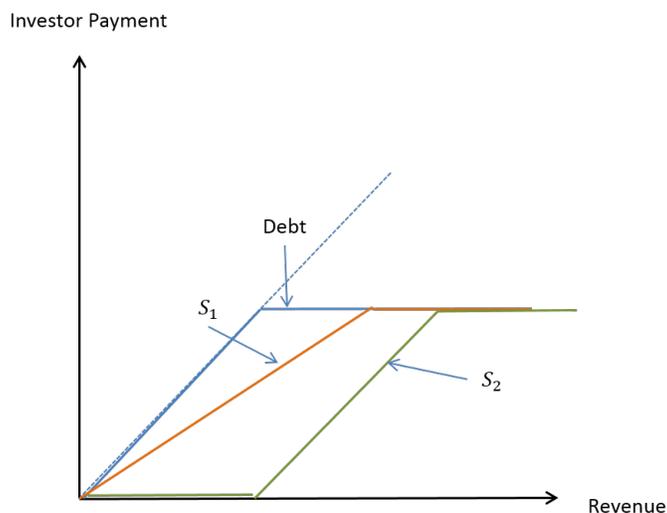
long as the security is a concave function of the joint proceeds, the firm will not benefit from the externalities.It can do no better than keep transactions independent and finance each project separately with debt. This arrangement is not only simple, but also robust. If degenerate outcomes are ruled out (see Remark), then this arrangement implements a bargaining game with a unique outcomes. The basic securities, debt and equity, are concave and securities which are convex combinations of them are concave as well. To implement a non-concave payment schedule, firms must use more sophisticated securities, such as option contracts, that are not always available.
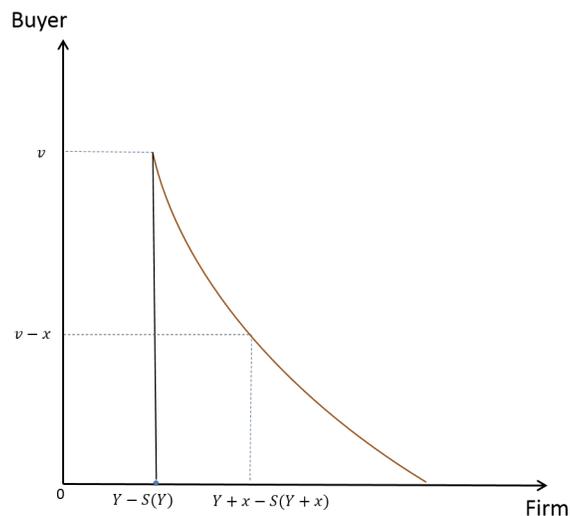
Remark:

5.1   Dynamic Bargaining

In this subsection we consider a strategic bargaining game and characterize the equilibrium (with refinements) for all concave securities.

The buyers arrive at the same time and the procedure goes as follows. The firm makes offers simultaneously and privately to each buyer. The buyers then simultaneously and independently either accept or reject. Any buyer who accepts pays and leaves. The game continues with the remaining buyers. These buyers simultaneously and independently make offers to the firm who chooses which, if any, to accept. There is a probability of breakdown between rounds: following any rejected offer, the game continues with probability p and break-down occurs with probability $1 - p$. The game ends if all goods are sold or a breakdown occurs. When the game ends, the proceeds from the sale are divided with the outside investor. The payoffs are given by $G(S)$ and $G(S_1, \ldots, S_N)$. Past transactions are observable but offers are private. The solution concept will be Perfect Bayesian Equilibrium (Fudenberg and Tirole (1991)).

- Several papers provide a strategic foundation for bilateral stability or the stronger concept of consistency in various bargaining situations (Krishna and Serrano (1996), Hart and Mas-Colell (1996), Collard-Wexler et al. (2014)). That is, each paper specifies a dynamic bargaining procedure and shows that the equilibria outcomes are close to (as offers become frequent converge to) the set of bilaterally stable or consistent outcomes. The latter two papers use refinements. These results do not apply to the bargaining problems in our model, we shortly explain the differences.

- Krishna and Serrano (1996), for instance, consider the standard pie division problem where partial agreements are possible. They show that a sequential proposer game has a unique subgame perfect equilibria. The key feature of this game is that if some players accept a division proposal, they will receive their portion of the pie, say Δ, and in the continuation game the remaining players will negotiate a division of the remaining pie $v - \Delta$.

  In our model, since values are independent, agreements with some buyers do not affect the surplus of others, but they may change the shape of the utility frontier in the continuation game; Figure 4 depicts the utility frontier in subgame with a single remaining buyer and the previous proceeds

Figure 4: Payoff frontier $B_j(S, Y)$

are Y. Additionally, the feasible payoff sets need not be convex gener-
ating multiple equilibria in bilateral subgames (see previous Remark...),
and as a result, the large game need not have a unique equilibrium even
with refinements. The arguments in the above papers are inductive and
rely on convexity.

For tractability, we will use equilibrium refinements: 1) Strategies are *sta-
tionary* in that actions depend only on the payoff relevant state variables,
the revenues from previous agreements and the buyers that have not reached
agreements; and 2) The off equilibrium beliefs are restricted. Since offers are
private, a buyer that receives an off equilibrium offer makes conjectures about
offers made to other players and the off equilibrium beliefs can have a large
effect on the continuation game generating multiplicity. A buyer has *passive
beliefs* if she believes that other buyers received their equilibrium price. The
need for such refinements is common when agreements among pairs of players
affect payoffs for other players (for instance in vertical contracting when there
are several downstream firms, e.g. McAfee and Schwartz (1994)).

24

Let $\hat{\mathcal{F}}G(S)$ be the set of agreements which are a limit, as $p \to 1$, of a PBE agreement with passive beliefs and stationary strategies.

Proposition. *If the bargaining solution is $\hat{\mathcal{F}}$ and securities are concave, then financing projects separately with debt is optimal.*

A proof is given in the appendix. The basic idea is to use the outcomes of bilateral subgames to bound the outcomes in the large game, we sketch out the key steps. Let $x_j(Y)$ denote the maximal equilibria price in a subgame where only buyer j remains, the previous proceeds are Y and the probability of breakdown vanishes. The main step shows if a sequence of equilibria outcomes (with passive beliefs and stationary strategies) converges to the outcome $y_1, \ldots, y_N$ as $p \to 1$, the equilibrium conditions imply

$$y_j \le x_j(Y_{-j}) \text{ where } Y_{-j} = \sum_{i \neq j}^{N} y_i, \ \forall j \tag{5}$$

The arguments is by induction on the number of buyers and the key point is to pin down how the price $x_j(Y)$ changes with the previous proceeds Y. Lemma 3 proves that all bilateral subgames satisfy a monotonicity property: the firm's maximal payoff $Y + x_j(Y)$ strictly increases with the previous proceeds Y.

This is not obvious. Figure 4 depicts the feasible payoff set $B_j(S,Y)$ in a bilateral subgame, observe that the previous proceeds Y determine the status quo payoff $Y - S(Y)$ and the shape of the utility frontier. The price $x_j(Y)$ may decrease with the previous proceeds Y, but we are able to bound this change, $\Delta > x_j(Y) - x_j(Y + \Delta)$ for $\Delta > 0$. For example, if the security is debt, the maximal equilibria price

$$x_j(Y) = \max \left[\frac{1}{2} (v_j + D - Y), \frac{1}{2}v_j \right]$$

This price weakly decreases with Y, but the firm's payoff $Y + x_j(Y)$ strictly increases.

Given the monotonicity property, a relatively straightforward, albeit tedious, line of arguments establishes Inequality (5) from the equilibria conditions.

Finally, since we know what happens in bilateral subgames, an upper bound on the firm's equilibria payoffs immediately follows.

6 Welfare implications

In our model, the investment I and the values $v_1, \ldots, v_N$ are given. Financial contracts may change how the social surplus $V - I$ is divided but do not affect welfare. This section presents two examples where financial contracts may have important welfare implications. In the first example, the investment level I is endogenous and in the second example, the firm decides how to allocate the output between the buyers, the distribution of values $v_1, \ldots, v_N$ is endogenous.

The focus is on standard debt contracts because they are prevalent. There are two main findings. First, in markets where hold-up problems reduce the potential gains from trade, debt-financing may strengthen the incentives to make ex-ante investments. Whether the hold-up problem is resolved depends mainly on the number of buyers. Second, debt may create inefficient distortions in ex-post allocations, influencing how much an exporter trades with local distributors for example.

6.1 Ex-ante investment and the hold-up problem

Suppose that a firm produces N units, each one for a specific buyer. The firm can make a costly investment, denoted by I, that increases the quality of the units. The buyers values are symmetric $v_i = v_j = v(I)$ where $v(0) = 0$, the function is increasing and concave. The firm chooses how much to invest, produces the units and then negotiates prices with the buyers.

The investment I generates surplus $V(I) = Nv(I)$ to the buyers and costs I to the firm, so the social surplus is $V(I) - I$. Denote the socially efficient investment level by $I^* = \arg\max V(I) - I$ and assume that $0 < I^* < \infty$. If the firm finances the investment on its own, the cost of investment is sunk and does not affect subsequent negotiations, each buyer will pay $\frac{v}{2}$, the firm will invest $I_0 = \arg\max \frac{V(I)}{2} - I$ and since v is concave, $I_0 < I^*$.

While outcomes with higher investments and higher prices are Pareto improving, opportunistic behavior make them infeasible. The point is well known. The firm bears the entire cost of investment and gets only a fraction of the gains. However, if the firm can extract a larger share of the surplus with external financing, it will have stronger incentives to invest. Whether debt contracts can achieve a Pareto improving outcome, depends mainly on the number of buyers.

To see it, suppose that the investment is financed with debt which repaid from the joint proceeds, so $S(X) = \min[X, D]$ where X are the proceeds. The firm will choose the investment I and the repayment D to maximize profits, investors at least break even, $D \geq I$. The bargaining outcomes are bilaterally stable.

Corollary. *When the investment is financed with debt, then either*

1) $I_0 > 0$, there exists a finite number of buyers $N_0 \geq 2$ such that for $N > N_0$ the firm invests I_0 and for $N_0 \geq N$ the firm invests I^*;

2) $I_0 = 0$, the firm invests I^*.

No generality is lost by restricting attention to $V(I) > D$. Recall that when the firm negotiates with N buyers and the values are v, each buyer paying

$$
x = \begin{cases} \frac{v}{2} & \text{if } (N-1)\frac{v}{2} \geq D \\ \frac{v+D}{N+1} & \text{if } (N-1)\frac{v}{2} < D \end{cases}
$$

is the maximal bilaterally stable outcome (Example 1). First, since each price decreases with the number of units, it should be verified that the firm indeed prefers to sell all the units. It is not hard to check that if the firm sells $m < N$ units, then the gains from selling another unit outweigh the loses on previous units.[16] Thus, given a debt of D and values v, the firm's profit is

_____

[16]Suppose that the firm sells $m < N$. If $(m-1)\frac{v}{2} < m\frac{v}{2} < D$, then the firm gains $\frac{v+D}{m+2}$ from selling an additional unit and loses $m(\frac{v+D}{m+1} - \frac{v+D}{m+2})$ on the previous m units. And if $(m-1)\frac{v}{2} < D \leq m\frac{v}{2}$ then the firm gains $\frac{v}{2}$ from the additional unit and loses $m(\frac{v+D}{m+1} - \frac{v}{2})$ on the previous m units. The final case is obvious.

$\pi = \max\left[\frac{V}{2} - D, \frac{1}{N+1}(V - D)\right]$. Second, since profits decrease in D, the firm optimally sets $D = I$ and chooses an investment level to maximize

$$\pi(I) = \max\left[\frac{V(I)}{2} - I, \frac{1}{N+1}(V(I) - I)\right]$$

If $I_0 > 0$, then $\frac{V(I_0)}{2} - I_0 > 0$ and there exists an $N_0 \geq 2$ such that

$$\frac{1}{N_0 + 1}(V(I^*) - I^*) \geq \frac{V(I_0)}{2} - I_0 > \frac{1}{N_0 + 2}(V(I^*) - I^*)$$

and we have that for all $N > N_0$, $\max_I \pi(I) = \frac{V(I_0)}{2} - I_0$ and for $N_0 > N$, $\max_I \pi(I) = \frac{1}{N+1}(V(I^*) - I^*)$.[17]

Notice that v is concave, so $I_0 > 0$ if and only if $v'_+(0) > 2$. Therefore, if $2 \geq v'_+(0)$, then $0 \geq \frac{V(I)}{2} - I$ for all I and $\max_I \pi(I) = \frac{1}{N+1}(V(I^*) - I^*)$ for all N. The firm will invest efficiently in this case.

That is, as long as the firm receives a fraction of the social surplus, it will invest efficiently, despite the externalities. But the more buyers there are, the smaller the firm's share and if the share is too small, the basic hold-up problem persists.

The above example may have important implications. Hold-up problems arise in markets where specific investments are made and parties can not commit to terms of trade. Under surprisingly general conditions, even explicit contracts between the parties can not curb the opportunistic behavior and the gains from trade are reduced (cite). Financial contracts alleviate this problem in some cases, but in many settings the basic problem may persist.

6.2 Ex-post allocation decisions

Consider an exporter that decides how to distribute some output Q between N markets. An allocation $(q_1, \ldots, q_N)$ satisfies $\sum_{i=1}^{N} q_i = Q$ and $q_i \geq 0$. In

---

[17]If it happens that $\frac{V(I_0)}{2} - I_0 = \frac{1}{N_0+1}(V(I^*) - I^*)$, then the firm is indifferent between I_0 or I^*.

each market, the exporter will negotiate with a local distributor. The value of distributor i is $v_i = v(q_i)$ where $v(0) = 0$, the function is increasing and concave.

The even allocation $(\frac{Q}{N}, \ldots, \frac{Q}{N})$ maximizes total surplus $\sum_{i=1}^{N} v(q_i)$. The exporter has a debt of D which is repaid from the proceeds of the sales and $Nv\left(\frac{Q}{N}\right) > D$. If the exporter divides the output evenly between the markets, she will negotiate with N buyers each with a value $v(\frac{Q}{N})$ and get

$$\pi(\frac{Q}{N}, \ldots, \frac{Q}{N}) = \max\left[\frac{V^*}{2} - D, \frac{1}{N+1}(V^* - D)\right]$$

where $V^* = Nv(\frac{Q}{N})$. If the total output is distributed to a single market, the exporter will negotiate with a single buyer with value $V(Q)$ and get

$$\pi(Q, 0, \ldots, 0) = \frac{1}{2}(v(Q) - D)$$

The trade-off here is clear. Due to bargaining externalities, the exporter will extract a larger share of the surplus by trading in fewer markets. But with diminishing marginal returns, uneven allocations generate less surplus.

If there are constant returns to scale (or relatively small returns), the exporter will prefer to trade in a single market because $v(Q) \approx Nv(\frac{Q}{N}) = V^*$. If v is strictly concave and the debt is either very low or very high, the exporter will prefer to distribute the output evenly (think of the cases $D = 0$ or $D > V(Q)$). For intermediate debt values, however, the exporter may be better-off trading in a single market. Specifically, we have that[18]

$$\frac{N-1}{N+1}v(Q) > D > V^* - v(Q) \implies \pi(Q, 0, \ldots, 0) > \pi(\frac{Q}{N}, \ldots, \frac{Q}{N}).$$

Of course, the allocation $(Q, 0, \ldots, 0)$ need not be optimal, but it demonstrates that debt and the resulting bargaining externalities may distort the allocation. This example has some potentially important implications, for instance, a small difference in transportation costs or tariffs between markets may lead to disparities in trading patterns.

---

[18]It is necessary that $2v(Q) > (N+1)v(\frac{Q}{N})$.

7 Conclusion

This paper address a basic problem that arises in various situations, how to optimally divide proceeds which accumulate from bargaining? The main message of our analysis is that a simple contract performs well. When bargaining is bilateral, the optimal contract is debt. When bargaining is multilateral, contracts may create a complex structure of entangled agreements, yet under some common conditions the firm can do no better than keep transactions independent and finance each project separately with a standard loan. The results do not rely on the details of a particular bargaining model.

The model makes a clear distinction between markets. If there is competition between buyers, the firm's financing decision will not affect market prices: debt, equity and self-financing are equivalent. However, in markets where buyers have some bargaining power, financing decisions do matter. Our model suggests that multiple investment should be separated from the firm when possible and that debt is (weakly) preferred to equity or self-financing. Cross-market comparisons may provide some empirical evidence that support or refute these hypotheses.

In practice, of course, there are a number of reasons why some investments are incorporated within the firm and other investments are undertaken separately. The debt over-hang problem is a first order consideration underlying this decision. The advantage of co-insurance and the cost of risk contamination is another. But project financing is common within certain industries, primarily resource extraction and manufacturing, and the present model offers one possible explanation for cross-industry variations.

The bargaining advantage of debt on the one hand and the externalities that attenuate this advantage on the other have broader economic implications. We demonstrated the effect of debt-financing when hold-up problems reduce the potential gains from trade. Debt may also distort ex-post allocation decisions, influencing the patterns of trade between an exporter and local distributors for example. These examples, and perhaps others, raise some interesting questions that may be worth pursuing in a more general framework.

References

Antic, N.: 2014, Contracting with unknown technologies, *unpublished paper, Princeton University* .

Axelson, U.: 2007, Security design with investor private information, *The Journal of Finance* **62**(6), 2587–2632.

Bolton, P. and Scharfstein, D. S.: 1990, A theory of predation based on agency problems in financial contracting, *The American Economic Review* pp. 93–106.

Brander, J. A. and Lewis, T. R.: 1986, Oligopoly and financial structure: The limited liability effect, *The American Economic Review* **76**(5), 956–970.

Bronars, S. G. and Deere, D. R.: 1991, The threat of unionization, the use of debt, and the preservation of shareholder wealth, *The Quarterly Journal of Economics* **106**(1), 231–54.

Chevalier, J. A.: 1995, Capital structure and product-market competition: Empirical evidence from the supermarket industry, *The American Economic Review* pp. 415–435.

Collard-Wexler, A., Gowrisankaran, G. and Lee, R. S.: 2014, Bargaining in bilateral oligopoly: An alternating offers representation of the "nash-in-nash" solution. Unpublished manscript.

Conley, J. P. and Wilkie, S.: 1991, The bargaining problem without convexity: extending the egalitarian and kalai-smorodinsky solutions, *Economics Letters* **36**(4), 365–369.

DeMarzo, P. M.: 2005, The pooling and tranching of securities: A model of informed intermediation, *Review of Financial Studies* **18**(1), 1–35.

DeMarzo, P. M., Kremer, I. and Skrzypacz, A.: 2005, Bidding with securities: Auctions and security design, *American Economic Review* **95**(4), 936–959.

Esty, B. C.: 2004, Why study large projects? an introduction to research on project finance, *European Financial Management* **10**(2), 213–224.

Fudenberg, D. and Tirole, J.: 1991, Perfect bayesian equilibrium and sequential equilibrium, *journal of Economic Theory* **53**(2), 236–260.

Gale, D. and Hellwig, M.: 1985, Incentive-compatible debt contracts: The one-period problem, *The Review of Economic Studies* **52**(4), 647–663.

Harsanyi, J. C.: 1963, A simplified bargaining model for the n-person cooperative game, *International Economic Review* **4**(2), 194–220.

Hart, O. and Moore, J.: 1997, Default and renegotiation: A dynamic model of debt, *Technical report*, National Bureau of Economic Research.

Hart, S. and Mas-Colell, A.: 1996, Bargaining and value, *Econometrica: Journal of the Econometric Society* pp. 357–380.

Herrero, M. J.: 1989, The nash program: non-convex bargaining problems, *Journal of Economic Theory* **49**(2), 266–277.

Innes, R. D.: 1990, Limited liability and incentive contracting with ex-ante action choices, *Journal of Economic Theory* **52**(1), 45 – 67.

Jensen, M. C. and Meckling, W. H.: 1979, *Theory of the firm: Managerial behavior, agency costs, and ownership structure*, Springer.

Kalai, E. and Smorodinsky, M.: 1975, Other solutions to nash's bargaining problem, *Econometrica* **43**(3), pp. 513–518.

Kaplan, S. N. and Strömberg, P.: 2009, Leveraged buyouts and private equity, *The Journal of economic perspectives* **23**(1), 121–146.

Krishna, V. and Serrano, R.: 1996, Multilateral bargaining, *The Review of Economic Studies* **63**(1), 61–80.

Lensberg, T.: 1988, Stability and the nash solution, *Journal of Economic Theory* **45**(2), 330–341.

Matsa, D. A.: 2010, Capital structure as a strategic variable: Evidence from collective bargaining, *The Journal of Finance* **65**(3), 1197–1232.

McAfee, R. P. and Schwartz, M.: 1994, Opportunism in multilateral vertical contracting: Nondiscrimination, exclusivity, and uniformity, *The American Economic Review* **84**(1), 210–230.

Myers, S. C. and Majluf, N. S.: 1984, Corporate financing and investment decisions when firms have information that investors do not have, *Journal of financial economics* **13**(2), 187–221.

Nash, J. F.: 1950, The bargaining problem, *Econometrica: Journal of the Econometric Society* pp. 155–162.

Rajan, R. G.: 1992, Insiders and outsiders: The choice between informed and arm's-length debt, *The Journal of Finance* **47**(4), 1367–1400.

Rubinstein, A.: 1982, Perfect equilibrium in a bargaining model, *Econometrica* **50**(1), 97–109.

Stole, L. A. and Zwiebel, J.: 1996a, Intra-firm bargaining under non-binding contracts, *The Review of Economic Studies* **63**(3), 375–410.

Stole, L. A. and Zwiebel, J.: 1996b, Organizational design and technology choice under intrafirm bargaining, *The American Economic Review* **86**(1), 195 – 222.

Townsend, R. M.: 1979, Optimal contracts and competitive markets with costly state verification, *Journal of Economic theory* **21**(2), 265–293.

Zhou, L.: 1997, The nash bargaining theory with non-convex problems, *Econometrica: Journal of the Econometric Society* pp. 681–685.

8   Appendix

Proposition. *If the bargaining solution is $\hat{\mathcal{F}}$ and securities are concave, then financing projects separately with debt is optimal.*

Let S be a concave security. To characterize the equilibrium payoffs, it is without the loss of generality to assume that $S' \leq 1$.[19] Also, we will assume that S is twice continuously differentiable. In a subgame game where only buyer j remains and the previous proceeds are Y, let $x_j^F(Y)$ and $x_j^B(Y)$ be the maximal SPE prices when the firm or buyer make the first offer respectively. Note that we can restrict attention to $Y \leq V - v_j$.

Lemma 1. *For $Y \in [0, V - v_j]$ and $p < 1$, the functions $x_j^F(Y), x_j^B(Y)$ are well defined; continuous; and $Y + x_j^B(Y)$ and $Y + x_j^F(Y)$ strictly increase.*

Proof. Consider a subgame where only buyer j remains and the previous proceeds are Y. Let $x_0(Y) = \sup\{x : S'(Y + x) = 1\}$. Note that S is concave and the firm's profit from an agreement x is $Y + x - S(Y + x)$, therefore, for $x \leq x_0(Y)$, the firm's profit is zero, and for price $x > x_0(Y)$, the firm's profit is positive and strictly increases in x.

First, there exists an SPE to this game. If $S'(Y + v_j) = 1$, then $v_j \leq x_0(Y)$ and any price $x \leq v_j$ can trivially be supported as an SPE (because the firm's payoff is zero). If $S'(Y + v_j) < 1$, we can find a price pair x_B, x_F where $x_0(Y) < x_B < x_F < v_j$ that solves,

$$
\begin{aligned}
v_j - x_F &= p(v_j - x_B) & (6) \\
Y + x_B - S(Y + x_B) &= (1 - p)(Y - S(Y)) + p(Y + x_F - S(Y + x_F)) & (7)
\end{aligned}
$$

To see it, let

$$
g(x, Y) = x - S(Y + x) + S(Y) - p\left(px + (1 - p)v_j - S(Y + px + (1 - p)v) + S(Y)\right)
$$

Notice that $g(x_0(Y), Y) < 0 < g(v_j, Y)$ and there exists a price x_B such that $g(x_B, Y) = 0$ and $x_B \in (x_0, v)$. Note that the firm's payoff strictly increases

---
[19]If for some Y, $S'(Y) > 1$ then for all $Y' < Y$, $S'(Y') > 1$ and the only possible equilibrium in this region is that each buyer pays 0.

with the price in this domain and it is straightforward to construct an SPE that supports the prices x_B and $x_F = (1-p)v + px_B$. [20]

The set of equilibria outcomes in non-empty, closed and bounded, so the maximal prices exist and the functions $x_j^B(Y), x_j^F(Y)$ are well defined. Moreover, when $S'(Y + v_j) < 1$, the maximal prices satisfy conditions (6) and (7). This follows from standard arguments: the maximal prices are achieved without delay, $v_j - x_j^F(Y) \leq p(v - x_j^B(Y))$ because the buyer will not accept a higher price and $p(v - x_j^B(Y)) \leq v_j - x_j^F(Y)$ because otherwise $x_F(Y)$ is not maximal. Similar argument establishes (7) on this domain.

Therefore,

$$x_j^B(Y) \quad = \quad \max\{x: \ g(x, Y) = 0 \text{ and } x \leq v_j\}$$

and $x_j^F(Y) = (1-p)v + px_j^B(Y)$. Note $x_j^B(Y) = v_j$ if and only if $S'(Y + v_j) = 1$.

To show continuity, take $Y \to Y_0$. First, if $S'(Y_0 + v_j) = 1$, then $x_j^B(Y_0) = v_j$ and $x_0(Y) \to v_j$. Since $x_0(Y) \leq x_j^B(Y) \leq v_j$, we have that $x_j^B(Y) \to v_j$. Second, if $S'(Y_0 + v_j) < 1$, $x_j^B(Y) \to x_j^B(Y_0)$ by the maximum theorem.

Finally, we establish that $Y + x_j^B(Y)$ strictly increases. This trivially follows when $Y \leq \max\{Y: S'(Y + v_j) = 1\}$ because $x_j^B(Y)$ is constant. For the case that $Y > \max\{Y: S'(Y + v_j) = 1\}$, observe that $x_j^B(Y)$ is implicitly defined by $g\left(x_j^B(Y), Y\right) = 0$. The following inequalities are true. First, in an ϵ-neighborhood of $(x_j^B(Y), Y)$,

$$\frac{\partial g}{\partial x} > \frac{\partial g}{\partial Y} \tag{8}$$

To see it,

$$\frac{\partial g}{\partial x} > \frac{\partial g}{\partial Y} \iff 1 + p > S'(Y) + pS'(Y + px + (1-p)v)$$

and for ϵ sufficiently small, $S'(Y + px + (1-p)v) < 1$ (because $x_0(Y) < x_j^B(Y)$). Second

---

[20]The strategies are stationary, the buyer always offers x_B and the firm x_F, each player only accepts an offer when the payoff is no less than the continuation payoff assuming her offer is accepted in the next period is an SPE.

$$\frac{\partial g}{\partial x}|_{x=x_j^B(Y)} \geq 0 \qquad (9)$$

because otherwise, $x_j^B(Y)$ is not the maximal solution to $g(x, Y) = 0$ (recall that $g(v_j, Y) > 0$). There are two options.

If the (9) holds with equality, $\frac{\partial g}{\partial x}|_{x=x_j^B(Y)} = 0$, then (8) implies that $\frac{\partial g}{\partial Y} < 0$ in the neighborhood around $(x_j^B(Y), Y)$. Therefore, for all $\Delta > 0$ sufficiently small, $g(x_j^B(Y), Y + \Delta) < 0$ and since $g(v, Y + \Delta) > 0$, it must be that $x_j(Y + \Delta) > x_j(Y)$. Thus, $Y + x_j^B(Y)$ strictly increases.

If the inequality in 9 is strict $\frac{\partial g}{\partial x}|_{x=x_j^B(Y)} > 0$, we can use the implicit function theorem,

$$\frac{dx_j^B}{dY} = -\frac{\frac{\partial g}{\partial Y}}{\frac{\partial g}{\partial x}} = -\frac{S'(Y) - S'(Y + x) - p\left(S'(Y) - S'(Y + px + (1 - p)v)\right)}{1 - S'(Y + x) - p^2\left(1 - S'(Y + px + (1 - p)v)\right)}$$

and since $\frac{\partial g}{\partial x} > \frac{\partial g}{\partial Y}$ and $\frac{\partial g}{\partial x}|_{x=x_j^B(Y)} > 0$, we have that

$$\frac{dx_j^B}{dY} > -1$$

and $Y + x_j^B(Y)$ strictly increases.

$\square$

Note that the maximal prices are defined for a given p, so we denote them by $x_{j,p}^B(Y)$ and $x_{j,p}^F(Y)$. In a subgame game where only buyer j remains and the previous proceeds are Y, let be the maximal $x_j(Y)$ be the maximal SPE prices when the probability of a breakdown vanishes.

Lemma 3. For $Y \in [0, V - v_j]$, $x_j(Y)$ is well defined and satisfies $x_j(Y) = \lim_{p \to 1} x_{j,p}^B(Y) = \lim_{p \to 1} x_{j,F}^B(Y)$.

Proof. Given $p < 1$, $x_{j,p}^B(Y)$ and $x_{j,p}^F(Y)$ are the maximal price in a subgame where only buyer j remains and the previous proceeds are Y. We will show that these prices converge and $\lim_{p \to 1} x_{j,p}^B(Y) = \lim_{p \to 1} x_{j,p}^F(Y)$.

If $S'(Y + v_j) = 1$ we are done, suppose that $S'(Y + v_j) < 1$. Let

$$g(x, Y, p) = x - S(Y + x) + S(Y) - p\left(px + (1 - p)v_j - S(Y + px + (1 - p)v) + S(Y)\right)$$

and

$$x_p = \max\{x : g(x, Y, p) = 0\} \tag{10}$$

From the previous lemma, $x_{j,p}^B(Y) = x_p$. The intermediate value theorem simplifies condition (10),

$$g(x, Y, p) = 0 \iff x + \frac{x + S(Y) - S(Y + x)}{p\left(1 - S'(Y + \tilde{x}_p)\right)} = v_j \tag{11}$$

where $x < \tilde{x}_p < px + (1 - p)v_j$. To see it,

$$
\begin{aligned}
S(Y + px + (1 - p)v) &= S'(Y + x) + S'(Y + \tilde{x}_p)(1 - p)(v - x) \\
\implies g(x, Y, p) &= (1 - p)\left\{x + S(Y) - S(Y + x) - p(v_j - x)\left(1 - S'(Y + \tilde{x}_p)\right)\right\}
\end{aligned}
$$

We therefore define

$$x_0 = \max\left\{x : x + \frac{x + S(Y) - S(Y + x)}{1 - S'(Y + x)} = v_j \text{ and } x \le v_j\right\} \tag{12}$$

Note that $0 < x_0 < v$ and $0 < x_p < v$ for all p. To establish convergence, we will show that

$$\liminf\{x_p\} \ge x_0 \ge \limsup\{x_p\}$$

First, given $\delta > 0$, it is true that $x_p > x_0 - \delta$ for all p sufficiently large. To see it, by continuity of conditions (12)(11), there exists $p_\delta < 1$ such that for each $p > p_\delta$ we can find an x where $|x - x_0| < \delta$ and $g(x, Y, p) = 0$. Therefore, for all $p > p_\delta$ it must be that $x_p \ge x > x_0 - \delta$ and $\liminf x_p \ge x_0$.

Second, towards a contradiction, suppose that $\limsup x_p > x_0$. Let $\Delta = \limsup x_p - x_0$, by continuity of (11) there exists $x_1 > \limsup x_p - \Delta$ such that

$$x_1 + \frac{x_1 + S(Y) - S(Y + x_1)}{1 - S'(Y + x_1)} = v_j$$

37

so $x_0 > x_1 > \limsup x_p - \Delta$.

Thus, the sequence $\{x_p\}$ converges. From the previous lemma, $x_{j,p}^F(Y) = (1-p)v + px_p$ and therefore

$$\lim x_{j,p}^B(Y) = \lim x_{j,p}^F(Y) = x_j(Y) = x_0$$

$\square$

Lemma 4. *Let $\epsilon > 0$, if the agreements $y_1, \ldots, y_N$ are achieved in an equilibrium with passive beliefs and stationary strategies and $S'(Y) < 1$, then $\forall i : y_i < x_{i,p}^B(Y_{-i}) + \epsilon$, for all $p < 1$ sufficiently large.*

Note that if $S'(Y) = 1$ the firm's profit is zero.

Proof. Let $\epsilon > 0$ and suppose that the agreements $y_1, \ldots, y_N$ are achieved in equilibrium. If the firm makes the offers, the argument is straightforward. In an equilibrium with passive beliefs, the firm's offers are accepted immediately. Since buyer j can reject the offer and guarantee a payoff of at least $p\left(v_j - x_j^B(Y_{-j})\right)$, it must be that

$$v_j - x_j^F(Y_{-j}) = p\left(v_j - x_j^B(Y_{-j})\right) \le v_j - y_j \implies y_j \le x_j^F(Y_{-j})$$

The first equality follows from standard arguments: the maximal prices are achieved without delay, $v_j - x_j^F(Y) \le p(v - x_j^B(Y))$ because the buyer will not accept a higher price and $p(v - x_j^B(Y)) \le v_j - x_j^F(Y)$ because otherwise $x_F(Y)$ is not maximal.

However, for subgames that begin with the buyers making offers, the argument is more intricate. First, if Buyer $j's$ offer is not accepted immediately, then by previous step $y_j \le x_j^F(Y_{-j})$, suppose the offer is accepted. Following a deviation by buyer j to a lower price, the firm will reject this and possibly other offers. Suppose that offers $y_1, \ldots, y_K$ are accepted and the others rejected, where $j > K$. From previous step, the game will end in the next period with price $y'_{K+1}, \ldots, y'_N$. Let $Y_K = \sum_{i=1}^{K} y_i$ and $Y' = Y_K + \sum_{i=K+1}^{N} y'_i$ and

38

$Y'_{-j} = Y' - y'_j$. The equilibrium conditions imply

$$p(v_j - y'_j) \leq v_j - y_j \tag{13}$$
$$Y \leq Y' + \epsilon \tag{14}$$

$$Y - S(Y) \leq (1 - p)(Y_K - S(Y_K)) + p(Y' - S(Y')) + \epsilon$$

Where (14) holds for all p sufficiently large. If either condition does not hold, there is a profitable deviation for buyer j (note that we do not assume that the equilibrium offers are accepted without delay).

From the previous step, $y'_j \leq x_j^F(Y'_{-j})$, and the equilibrium conditions (13) and (14) can be rewritten,

$$y_j \leq (1 - p)v + px_j^F(Y'_{-j}) \tag{15}$$
$$Y \leq Y' + \epsilon \leq Y'_{-j} + x_j^F(Y'_{-j}) + \epsilon \tag{16}$$

First, if $Y'_{-j} \leq Y_{-j}$, then Lemma 2 implies that $Y'_{-j} + x_j^F(Y'_{-j}) < Y_{-j} + x_j^F(Y_{-j})$ and together with (14), we have that

$$Y \leq Y'_{-j} + x_j^F(Y'_{-j}) + \epsilon < Y_{-j} + x_j^F(Y_{-j}) + \epsilon \implies y_j < x_j^F(Y_{-j}) + \epsilon$$

Second, suppose that $Y_{-j} < Y'_{-j}$. Given $\delta_1 > 0$, it must be that $Y' < Y + \delta_1$ for all $p < 1$ sufficiently large. Otherwise, since strategies are stationary, the firm would have a profitable deviation to reject K offers. To see it, notice that the firm will get $(1 - p)(Y_K - S(Y_K)) + p(Y' - S(Y'))$ from doing so and

$$Y - S(Y) < (1 - p)(Y_K - S(Y_K)) + p(Y' - S(Y'))$$

if $Y + \delta_1 < Y'$ and p is sufficiently large. Thus, since x_j^F is continuous, $x_j^F(Y'_{-j}) - x_j(Y_{-j}) < \frac{\epsilon}{2}$ for all $p < 1$ sufficiently large and together with condition (16) we have that $y_j \leq (1 - p)v + px_j(Y_{-j}) + p\frac{\epsilon}{2}$.

$\square$

Proposition 2 follows from Lemma 4.

Proof. Let $\boldsymbol{y^*} = (y_1^*, \ldots, y_N^*) \in \hat{\mathcal{F}}G(S)$, then there exists a sequence of price vectors $\{\boldsymbol{y_p}\} = \{(y_{1,p}, \ldots, y_{N,p})\}$ such that for all p, $\boldsymbol{y_p}$ is an equilibrium (with passive beliefs and stationary strategies) price vector and $\boldsymbol{y_p} \to \boldsymbol{y^*}$ as $p \to 1$ (point-wise).

First, we show that

$$y_j^* - S(Y^*) + S(Y_{-j}^*) \leq v_j - y_j^* \tag{17}$$

From Lemma 4,

$$y_{j,p} \leq x_{j,p}^B(Y_{-j,p}) + \epsilon_p \tag{18}$$

and $\epsilon_p \to 0$ as $p \to 1$.

In a continuation game where only buyer i remains, from Lemma 1, the firm gains from trade are no more than the buyers,

$$x_{j,p}^B(Y_{-j,p}) - S\left(Y_{-j,p} + x_{j,p}^B(Y_{-j,p})\right) + S(Y_{-j,p}) \leq v_j - x_{j,p}^B(Y_{-j,p}) \tag{19}$$

Since $S' \leq 1$, conditions (18) and (19) imply,

$$y_{j,p} - S(Y_{-j,p} + y_{j,p}) + S(Y_{-j,p}) \leq v_j - y_{j,p} + \epsilon_p \tag{20}$$

and taking the limit in (20) implies (17).

Second, since S is concave, $S(Y^*) - S(Y_{-j}^*) \leq \frac{y_j^*}{Y^*}(S(Y^*) - S(0))$ together with (17) implies

$$y_j^* - \left(\frac{y_j^*}{Y^*}(S(Y^*) - S(0))\right) \leq v_j - y_j^*$$

and in sum,

$$Y^* - S(Y^*) \leq V - Y^*$$

Finally, as long as $S(Y^*) \geq I$, the firm will not get more than $\frac{1}{2}(V - I)$. $\quad\square$

8.1 Other Bilateral Bargaining Models

In this subsection, we consider the axiomatic bargaining solutions of Nash (1950) and Kalai and Smorodinsky (1975). More precisely, we will use the natural extension of these solutions to problems with non-convex utility sets[21]. We present non-formal geometric argument for the optimality of debt.

First, suppose that the agreements in $\mathcal{FG}(S)$ maximize the product of the players utilities over the feasible payoff set. Notice that the solution is well-defined but need not always be unique. In Figure 1, observe that as long as $S' \geq 0$, the slope of the utility frontier is at most -1. Therefore, the agreements that maximize the Nash product lay above the $45°$ line. That is, the firm will not get more than the buyer (Lemma 1 holds) and $\frac{1}{2}(v - I)$ is an upper bound on the firm's payoff.

In the bargaining game with debt, there is unique agreement which maximizes the Nash product, $x = \frac{v+D}{2}$, and this agreement achieves the firm's maximal payoff.

Second, consider the solution of Kalai and Smorodinsky (1975). This solution is monotonic which implies that if a security S implements an agreement x_0 and $S'(x_0) \geq 0$, then security

$$
\hat{S}(x) = \begin{cases} S(x) & \text{if } x \leq x_0 \\ S(x_0) & \text{if } x > x_0 \end{cases}
$$

implements an agreement price x_1 where $x_1 \geq x_0$. Securities S and $\hat{S}$ are depicted in Figure **??**. The firm is weakly better-off with security $\hat{S}$ while the investors no worse off, because $\hat{S}(x_1) = S(x_0)$. A debt security implements the same outcome as security $\hat{S}$ and therefore weakly dominates any other increasing security.

....figure

---

[21]Several papers extend these solutions to non-convex problems, see for example Zhou (1997) and Conley and Wilkie (1991).

8.2 More general contracts

The analysis focused two types of contracts, joint and separate, but more general contracts can also be considered. For one, the firm should be able to finance only a subset of projects separately and the others jointly. It is clear that no generality was lost in focusing on the two extreme partitions above. However, more general contracts could be written if securities would depend on the vector of revenues instead of the sum. Our analysis can be easily extended to this case.

Suppose that a security $S : \mathbb{R}^N \to \mathbb{R}_+$ specifies the payment to investors as a function of the proceeds from the sales. That is, following the agreements over prices $x_1, \ldots, x_N$, the firm will pay $S(x_1, \ldots, x_N)$ to investors and keep the remainder $X - S(x_1, \ldots, x_N)$.

Note that these securities rely on a stronger, and perhaps unrealistic, assumption that the contract can distinguish the proceeds from different transactions. If multiple projects are financed within the firm and the firm can shuffle the proceeds from one project to another, then the payment to investors would only depend on the sum of the proceeds.

Bold symbols denote vectors. Given a price vector $\boldsymbol{x} = x_1, \ldots, x_N$, let $\hat{\boldsymbol{x}}_j = (x_1 \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_N)$ denote the vector that results in replacing the j-th element of $\boldsymbol{x}$ with 0. The marginal payment to investors from reaching an agreement with buyer j is $S(\boldsymbol{x}) - S(\hat{\boldsymbol{x}}_j)$. We assume that securities are sub-additive in the following sense: the sum of the marginal payments does not exceed the total payment,

$$\sum_{i=1}^N S(\boldsymbol{x}) - S(\hat{\boldsymbol{x}}_i) \leq S(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \geq 0 \tag{21}$$

Proposition. *When securities satisfy (21), financing projects separately with debt is optimal.*

The argument is identical to the proof of Lemma 2.

Proof. First, if the prices $x_1, \ldots, x_N$ are bilaterally stable, then the firm's gain

from each trade will not exceed the buyer's gain,

$$\forall i : \ x_i - (S(\boldsymbol{x}) - S(\hat{\boldsymbol{x}_i})) \leq v_i - x_i$$

Second, if the security satisfies (21), the firm's profit will not exceed the sum of the firm's gains from each trade,

$$X - S(\boldsymbol{x}) \leq \sum_{i=1}^{N} x_i - (S(\boldsymbol{x}) - S(\hat{\boldsymbol{x}_i}))$$

and we have that the firm's total payoff will not exceed the buyers',

$$X - S(X) < V - X$$

$\square$