Contents

Lecture 4

Public Goods and Private Goods

One Public Good, One Private Good

Cecil and Dorothy[1] are roommates, too. They are not interested in card games or the temperature of their room. Each of them cares about the size of the flat that they share and the amount of money he or she has left for "private goods". Private goods, like chocolate or shoes, must be consumed by one person or the other, rather than being jointly consumed like an apartment or a game of cards. Cecil and Dorothy do not work, but have a fixed money income $W. This money can be used in three different ways. It can be spent on private goods for Cecil, on private goods for Dorothy, or it can be spent on rent for the apartment. The rental cost of a flat is $c per square foot.

Let X_C and X_D be the amounts that Cecil and Dorothy, respectively, spend on private goods. Let Y be the number of square feet of space in the flat. The set of possible outcomes for Cecil and Dorothy consists of all those triples, (X_C, X_D, Y) that they can afford given their wealth of $W. This is just the set:

$$\{(X_C, X_D, Y) | X_C + X_D + cY \leq W\}$$

In general, Cecil's utility function might depend on Dorothy's private con-

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[1]When I first produced these notes, the protagonists were named Charles and Diana. Unfortunately many readers confused these characters with the Prince of Wales and his unhappy spouse. Since it would plainly be in bad taste to suggest, however inadvertently, that members of the royal family might have quasi-linear preferences, two sturdy commoners, Cecil and Dorothy, have replaced Charles and Diana in our cast of characters.

sumption as well as on his own and on the size of the apartment. He might, for example, like her to have more to spend on herself because he likes her to be happy. Or he might be an envious lout who dislikes her having more to spend than he does. Thus, in general, we would want him to have a utility function of the form:

$$U_C(X_C, X_D, Y).$$

But for our first pass at the problem, let us simplify matters by assuming that both Cecil and Dorothy are totally selfish about private goods. That is, neither cares how much or little the other spends on private goods. If this is the case, then their utility functions would have the form:

$$U_C(X_C, Y) \quad \text{and} \quad U_D(X_D, Y).$$

With the examples of Anne and Bruce and of Cecil and Dorothy in mind, we are ready to present a general definition of public goods and of private goods. We define a public good to be a social decision variable that enters simultaneously as an argument in more than one person's utility function. In the tale of Anne and Bruce, both the room temperature and the number of games of cribbage were public goods. In the case of Cecil and Dorothy, if both persons are selfish, the size of their flat is the only public good. But if, for example, both Cecil and Dorothy care about Dorothy's consumption of chocolate, then Dorothy's chocolate would by our definition have to be a public good.

Perhaps surprisingly, the notion of a "private good" is a more complicated and special idea than that of a public good. In the standard economic models, private goods have two distinguishing features. One is the distribution technology. For a good, say chocolate, to be a private good it must be that the total supply of chocolate can be partitioned among the consumers in any way such that the sum of the amounts received by individuals adds to the total supply available.[2] The second feature is selfishness. In the standard models of private goods, consumers care only about their own consumptions of any private good and not about the consumptions of private goods by others.

In the story of Cecil and Dorothy, we have one public good and one private good.[3] To fully describe an allocation of resources on the island we

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[2]It is possible to generalize this model by adding a more complicated distribution technology and still to have a model which is in most respects similar to the standard private goods model.

[3]According to a standard result from microeconomic theory, we can treat "money for private goods" as a single private good for the purposes of our model so long as prices of private goods relative to each other are held constant in the analysis.

need to know not only the total output of private goods and of public goods, but also how the private good is divided between Cecil and Dorothy. The allocation problem of Cecil and Dorothy is mathematically more complicated than that of Anne and Bruce. There are three decision variables instead of two and there is a feasibility constraint as well as the two utility functions. Therefore it is more difficult to represent the whole story on a graph. It is, however, quite easy to find interesting conditions for Pareto optimality using Lagrangean methods. In fact, as we will show, these conditions can also be deduced by a bit of careful "literary" reasoning.

We begin with the Lagrangean approach. At a Pareto optimum it should be impossible to find a feasible allocation that makes Cecil better off without making Dorothy worse off. Therefore, Pareto optimal allocations can be found by setting Dorothy at an arbitrary (but possible) level of utility, $\bar{U}_D$ and maximizing Cecil's utility subject to the constraint that $U_D(X_D, Y) \geq \bar{U}_D$ and the feasibility constraint. Formally, we seek a solution to the constrained maximization problem: Choose X_C, X_D and Y to maximize $U_C(X_C, Y)$ subject to:

$$U_D(X_D, Y) \geq \bar{U}_D \quad \text{and} \quad X_C + X_D + cY \leq W.$$

We define an *interior Pareto optimum* to be an allocation in the interior of the set of feasible allocations. With the technology discussed here, an interior Pareto optimum is a Pareto optimal allocation in which each consumer consumes a positive amount of public goods and where the amount of public goods is positive. The Lagrangean for this problem is: are

$$U_C(X_C, Y) - \lambda_1 \left(\bar{U}_D - U_D(X_D, Y) \right) - \lambda_2 \left(X_C + X_D + cY - W \right) \quad (4.1)$$

A necessary condition for an allocation $(\bar{X}_C, \bar{X}_D, \bar{Y})$ to be an interior Pareto optimum is that the partial derivatives of the Lagrangean are equal to zero. Thus we must have:

$$\frac{\partial U_C(\bar{X}_C, \bar{X}_D)}{\partial X_C} - \lambda_2 = 0 \quad (4.2)$$

$$\lambda_1 \frac{\partial U_D(\bar{X}_C, \bar{X}_D)}{\partial X_D} - \lambda_2 = 0 \quad (4.3)$$

$$\frac{\partial U_C(\bar{X}_C, \bar{X}_D)}{\partial Y} + \lambda_1 \frac{\partial U_D(\bar{X}_C, \bar{X}_D)}{\partial Y} - \lambda_2 c = 0 \quad (4.4)$$

From 4.2 it follows that:

$$\lambda_2 = \frac{\partial U_C(\bar{X}_C, \bar{X}_D)}{\partial X_C}. \quad (4.5)$$

From 4.3 and 4.5 it follows that:

$$\lambda_1 = \frac{\partial U_C(\bar{X}_C, \bar{X}_D)}{\partial X_C} \div \frac{\partial U_D(\bar{X}_C, \bar{X}_D)}{\partial X_D} \tag{4.6}$$

Use 4.6 and 4.5 to eliminate λ_1 and λ_2 from Equation 4.4. Divide the resulting expression by $\frac{\partial U_C}{\partial X_C}$ and you will obtain:

$$\frac{\frac{\partial U_C(\bar{X}_C, \bar{X}_D)}{\partial Y}}{\frac{\partial U_C(\bar{X}_C, \bar{X}_D)}{\partial X_C}} + \frac{\frac{\partial U_D(\bar{X}_C, \bar{X}_D)}{\partial Y}}{\frac{\partial U_D(\bar{X}_C, \bar{X}_D)}{\partial X_D}} = c \tag{4.7}$$

This is the fundamental "Samuelson condition" for efficient provision of public goods. Stated in words, Equation 4.7 requires that the *sum* of Cecil's and Dorothy's marginal rates of substitution between flat size and private goods must equal the cost of an extra unit of flat relative to an extra unit of private goods.

Let us now try to deduce this condition by literary methods. The rate at which either person is willing to exchange a marginal bit of private consumption for a marginal increase in the size of the flat is just his marginal rate of substitution. Thus the left side of Equation 4.7 represents the amount of private expenditure that Cecil would be willing to give up in return for an extra foot of space plus the amount that Dorothy is willing to forego for an extra foot of space. If the left side of 4.7 were greater than c, then they could both be made better off, since the total amount of private expenditure that they are willing to give up for an extra square foot of space is greater than the total amount, c, of private good that they would have spend to get an extra square foot. Similarly if the left hand side of 4.7 were less than c, it would be possible to make both better off by renting a smaller flat and leaving each person more money to spend on private goods. Therefore an allocation can be Pareto optimal only if Equation 4.7 holds.

An Intriguing, but Generally Illegal Diagram

Economists love diagrams. Indeed, if there were an economists' coat of arms, it would surely contain crossing supply and demand curves.[4] When we discussed the special case where Cecil and Dorothy have quasilinear utility, we were able to use a diagram which is just as pretty as a supply and demand diagram, but tantalizingly different.

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[4]Perhaps also an Edgeworth box and an IS-LM diagram.

Recall that in the theory of private, competitive markets, we add individual demand curves "horizontally" to obtain the total quantity of a good demanded in the economy at any price. Equilibrium occurs at the price and quantity at which the aggregate demand curve crosses the aggregate supply curve. In Figure 4.1, the curves $M_1 M_1$ and $M_2 M_2$ are the demand curves of consumers 1 and 2 respectively and the horizontal sum is the thicker kinked curve. Where the line SS is the supply curve, equilibrium occurs at the point E.

Figure 4.1: **With Private Goods, Demand Curves Add Horizontally**
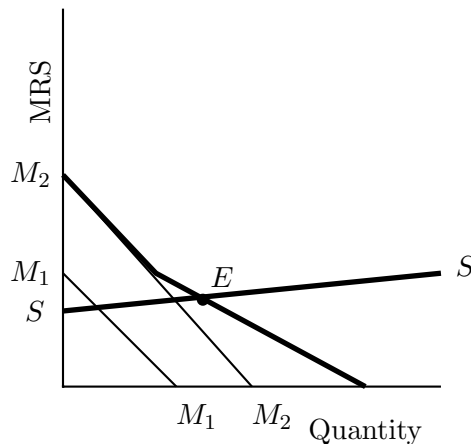
Howard Bowen [2] in a seminal article[5] that was published in 1943 (ten years before Samuelson's famous contribution) suggested a diagram in which one finds the efficient amount of public goods by summing marginal rate of substitution curves *vertically*. A closely related construction can also be found in a much earlier paper by a Swedish economist, Erik Lindahl [3]. We will look at Lindahl's diagram later. Lindahl and Bowen both realized that a Pareto efficient supply of public goods requires that the sum of marginal rates of substitution between the public good and the private good equals the marginal rate of transformation.

Bowen proposed the diagram that appears in Figure 4.2, where the amount of public good is represented on the horizontal axis and marginal rates of substitution are shown on the vertical axis. The curve $M_i M_i$ represents consumer i's marginal rate of substitution as a function of the

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[5]This article is best known for its pioneering contribution to the theory of voting and public choice.

Figure 4.2: **With Public Goods, MRS Curves Add Vertically**
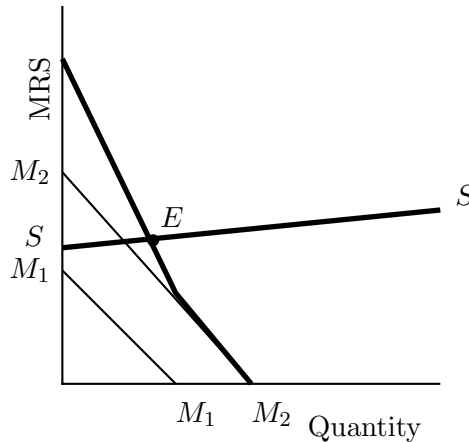
amount of public good. These curves are added vertically to find the sum
of marginal rates of substitution curve. The intersection E between the
summed marginal rate of substitution curve and the marginal rate of trans-
formation curve SS occurs at the Pareto optimal quantity for of public
goods.

Now the problem with Figure 4.2 is that in general, we have no right
to draw it, at least not without further assumptions. The reason is that
a person's marginal rate of substitution between public and private goods
depends in general not only on the amount of public goods available but
also on the amount of private goods he consumes. While in general we
have no right to draw these curves, we will show that we can draw them
if utility functions are quasilinear. This is the case because when utility is
quasilinear, an individual's marginal rate of substitution between public and
private goods depends only on the amount of public goods.

The fact that we cannot, in general, draw marginal rate of substitution
curves without making some assumption about the distribution of private
goods was first brought clearly to the attention of the profession in Samuel-
son's famous 1954 article. To be fair to Bowen, he seems to have been aware
of this problem, since he remarks that these curves can be drawn "assum-
ing a correct distribution of income"[6] though he did not discuss this point
as clearly as one might wish. Even without quasilinear utility, it would be

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[6]Notice the similarity to Musgrave's later suggestion that each branch of government
assume that the others are doing the right thing.

possible to draw marginal rate of substitution curves if we made an explicit assumption about how private consumption will be allocated contingent on the level of public expenditure.

A Family of Special Cases – Quasilinear Utility

Suppose that Cecil and Dorothy have utility functions that have the special functional form:

$$U_C(X_C, Y) = X_C + f_C(Y) \qquad (4.8)$$

$$U_D(X_D, Y) = X_D + f_D(Y) \qquad (4.9)$$

where the functions f_C and f_D have positive first derivatives and negative second derivatives.[7] This type of utility function is said to be *quasilinear.*

Since $\frac{\partial U^C}{\partial X^C} = 1$, it is easy to see that Cecil's marginal rate of substitution between the public and private goods is simply $\frac{\partial U_C}{\partial Y} = f'_C(Y)$. Likewise Dorothy's marginal rate of substitution is just $f'_D(Y)$. Therefore the necessary condition stated in equation 4.7 takes the special form

$$f'_C(Y) + f'_D(Y) = c \qquad (4.10)$$

Since we have assumed that f''_C and f''_D are both negative, the left side of (8) is a decreasing function of (Y). Therefore, given c, there can be at most one value of Y that satisfies Equation 4.10. Equilibrium is neatly depicted in the Figure 4.3.

The curves $f'_C(Y)$ and $f'_D(Y)$ represent Cecil's and Dorothy's marginal rates of substitution between public and private goods. (In the special case treated here, marginal rates of substitution are independent of the other variables X_C and X_D.) The curve $f'_C(Y) + f'_D(Y)$ is obtained by summing the individual m.r.s. curves *vertically* (rather than summing horizontally as one does with demand curves for private goods). The only value of Y that satisfies condition 4.10 is Y^*, where the summed m.r.s. curve crosses the level c.

As we will show later, the result that the optimality condition 4.10 uniquely determines the amount of public goods depends on the special

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[7]The assumptions about the derivatives of f_C and f_D ensure that indifference curves slope down and are convex toward the origin. The assumption that $U(X_C, Y)$ is linear in X_C implies that if the indifference curves are drawn with X on the horizontal axis and Y on the vertical axis, each indifference curve is a horizontal translation of any other, making the indifference curves are parallel in the sense that for given Y, the slope of the indifference curve at (X, Y) is the same for all X.

Figure 4.3: Summing Marginal Rates of Substitution
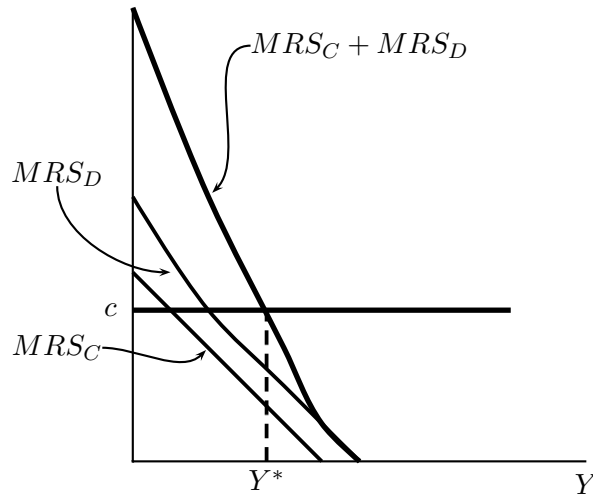
kind of utility function that we chose. The class of models that has this property is so special and so convenient that it has earned itself a special name. Models in in which everybody's utility function is linear in some commodity are said to have *quasilinear utility*.

As we have seen, when there is quasilinear utility, there is a unique Pareto optimal amount, Y^* of public good corresponding to allocations in which both consumers get positive amounts of the private good. This makes it very easy to calculate the utility possibility frontier. Since the supply of public good is Y^*, we know that along this part of the utility possibility frontier, Cecil's utility can be expressed as $U_C(X_C, Y^*) = X_C + f_C(Y^*)$ and Dorothy's utility is $U_D(X_D, Y^*) = X_D + f_D(Y^*)$. Therefore it must be that on the utility possibility frontier, $U_C + U_D = X_C + X_D + f_C(Y^*) + f_D(Y^*)$. After the public good has been paid for, the amount of the family income that is left to be distributed between Cecil and Dorothy is $W - p_Y Y^*$. So it must be that $U_C + U_D = W - p_Y Y^* + f_C(Y^*) + f_D(Y^*)$. Since the right side of this expression is a constant, the part of the utility possibility frontier that is achievable with positive private consumptions for both persons will be a straight line with slope -1.
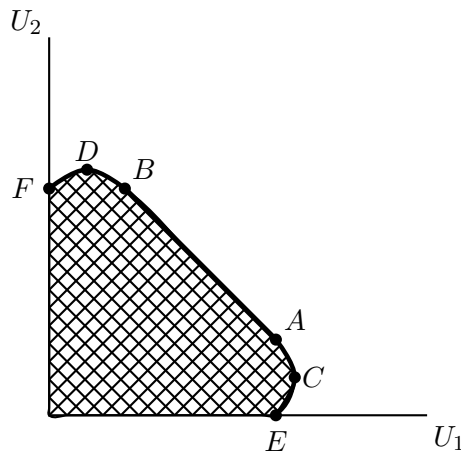
Example 4.1

Suppose that Persons 1 and 2 have quasilinear utility functions $U_i(X,Y) = X_i + \sqrt{Y}$. Public goods can be produced from private goods at a cost of 1 unit of private goods per unit of public goods and there are initially 3 units of private goods which can either be used to produce public goods or can be distributed between persons 1 and 2. Thus the set of feasible allocations is $\{(X_1, X_2, Y) \geq 0 | X_1 + X_2 + Y \leq 3\}$. The sum of utilities is $U_1(X_1, Y) + U_2(X_2, Y) = X_1 + X_2 + 2\sqrt{Y}$ which is equal to $X + 2\sqrt{Y}$ where $X = X_1 + X_2$. We will maximize the sum of utilities by maximizing $X + 2\sqrt{Y}$ subject to $X + Y \leq 3$. The solution to this constrained maximization problem is $Y = 1$ and $X = 3$. Any allocation $(X_1, X_2, 1) \geq 0$ such that $X_1 + X_2 = 2$ is a Pareto optimum.

In Figure 4, we draw the utility possibility set. We start by finding the utility distributions that maximize the sum of utilities and in which $Y = 1$ and $X_1 + X_2 = 2$. At the allocation $(2, 0, 1)$, where Person 1 gets all of the private goods, we have $U_1 = 2 + 1 = 3$ and $U_2 = 0 + 1 = 1$. This is the point A. If Person 2 gets all of the private goods, then $U_1 = 0 + 1 = 1$ and $U_2 = 2 + 1 = 3$. This is the point B. Any point on the line AB can be achieved by supplying 1 unit of public goods and dividing 2 units of private goods between Persons 1 and 2 in some proportions.

Figure 4.4: **A Utility Possibility Set**

Now let's find the Pareto optimal points that do not maximize the sum of utilities. Consider, for example, the point that maximizes Person 1's

utility subject to the feasibility constraint $X_1 + X_2 + Y = 3$. Since Person 1 has no interest in Person 2's consumption, we will find this point by maximizing $U_1(X_1, Y) = X_1 + \sqrt{Y}$ subject to $X_1 + Y = 3$. This is a standard consumer theory problem. If you set the marginal rate of substitution equal to the relative prices, you will find that the solution is $Y = 1/4$ and $X_1 = 2\frac{3}{4}$. With this allocation, $U_1 = 3\frac{1}{4}$ and $U_2 = 1/2$. This is the point C on Figure 4. Notice that when Person 1 controls all of the resources and maximizes his own utility, he still leaves some crumbs for Person 2, by providing public goods though he provides them from purely selfish motives. The curved line segment CA comprises the utility distributions that result from allocations $(W - Y, 0, Y)$ where Y is varied over the interval $[1/2, 1]$. An exactly symmetric argument will find the segment DB of the utility possibility frontier that corresponds to allocations in which Person 2 gets no private goods.

The utility possibility frontier, which is the northeast boundary of the utility possibility set, is the curve $CABD$. There are also some boundary points of the utility possibility set that are not Pareto optimal. The curve segment CE consists of the distributions of utility corresponding to allocations $(W - Y, 0, Y)$ where Y is varied over the interval $[0, 1/2]$. At these allocations, Person 1 has all of the private goods and the amount of public goods is less than the amount that Person 1 would prefer to supply for himself. Symmetrically, there is the curve segment DF in which Person 2 has all of the private goods and the amount of public goods is less than $1/2$. Finally, every utility distribution in the interior of the region could be achieved by means of an allocation in which $X_1 + X_2 + Y < 3$.

In this example, we see that at every Pareto optimal allocation in which each consumer gets a positive amount of private goods the amount of public goods must be $Y = 1$, which is the amount that maximizes the sum of utilities.

Generalizations

For clarity of exposition we usually discuss the case where a single public good is produced from private goods at a constant unit cost c. In this case, the set of feasible allocations consists of all $(X_1, \ldots, X_n, Y) \geq 0$ such that $\sum_i X_i + cY = W$ where W is the initial allocation of public goods. Most of our results extend quite easily to a more general technology in which the cost of production of public goods is not constant but where there is a convex production possibility set. For this more general technology,

the set of feasible allocations consists of all $(X_1, \ldots, X_n, Y) \geq 0$ such that $(\sum_i X_i, Y) \in T$ where T is a convex set, specifying all feasible combinations of gotal output of private goods and total amount of public goods produced.

Almost all of our results also extend in a pretty straightforward way to cases where there are many public goods, many private goods, and many people. I don't need to present the story here because the details of how this works are nicely sketched out in Samuelson's classic paper "The Pure Theory of Public Expenditure" [4], which is on your required reading list.

Discrete Choice and Public Goods

Some decisions about public goods are essentially discrete. For example, Cecil and Dorothy may choose whether or not to buy a car or whether or not to move from London to Chicago. For this kind of choice, the public goods setup is useful even though Samuelsonian calculations of marginal rates of substitution are not relevant.

Let X_C and X_D denote private consumption for Cecil and Dorothy, and let $Y = 0$ if they don't have a car and $Y = 1$. Assume that in initially, they have no car and their private consumptions are X_C^0 and X_D^0. Let us define Cecil's willingness to pay for a car as the greatest reduction in private consumption that he would accept in return for having a car. Similarly for Dorothy. In general, their willingnesses to pay for a car will depend on their initial consumptions if they don't buy a car. In particular Person i's willingness to pay for a car is the function $V_i(\cdot)$ is defined to be the solution to the equation $U_i(X_i^0 - V_i(X_i^0), 1) = U_i(X_i^0, 0)$. Suppose that the cost of the car is c. It will be possible for Cecil and Dorothy to achieve a Pareto improvement by buying the car if and only if the sum of their willingnesses to pay for the car exceeds the cost of the car. If this is the case, then each could pay for the car with each of them paying less than his or her willingness to pay. Therefore both would be better off.

Appendix: When Samuelson Conditions are Sufficient

So far, we have shown only that the Samuelson conditions are necessary for an interior Pareto optimum. The outcome is exactly analogous to the usual maximization story in calculus. For continuously differentiable functions, the first-order conditions are necessary for either a local or global maximum, but they are not in general sufficient. As you know, in consumer theory, if

utility functions are quasi-concave and continuously differentiable, then a consumption bundle that satisfies an individual's first-order conditions for constrained maximization will actually be a maximum for the constrained optimization problem. For the Samuelson conditions, we have the following result.

Proposition 1 (Sufficiency of Samuelson conditions) *If all individuals have continuously differentiable, quasi-concave utility functions $U(X_i, Y)$ which are strictly increasing in X_i and if the set of possible allocations consists of all $(X_1, \ldots, X_n, Y)$ such that $\sum X_i + cY = W$, then the feasibility condition $\sum \bar{X}_i + c\bar{Y} = W$ together with the Samuelson condition*

$$\sum_{i=1}^{n} \frac{\frac{\partial U_i(\bar{X}_i, \bar{Y})}{\partial Y}}{\frac{\partial U_i(\bar{X}_i, \bar{Y})}{\partial X_i}} = c \tag{4.11}$$

is sufficient as well as necessary for an interior allocation $(\bar{X}_1, \ldots, \bar{X}_n, \bar{Y}) >> 0$ to be Pareto optimal.

To prove Proposition 1, we need the following lemma, which is in general a useful thing for economists to know about. A proof of the lemma is straightforward and can be found, for example, in Blume and Simon's text *Mathematics for Economists* [1]. A function with the properties described in Lemma 1 is called a *pseudo-concave* function.

Lemma 1 *If the function $U(X, Y)$ is continuously differentiable, quasi-concave, and defined on a convex set and if at least one of the partial derivatives of U is non-zero at every point in the domain, then if (X, Y) is in the interior of the domain of U, then for any (X', Y'),*

- *if $U(X', Y') > U(X, Y)$ then*

$$\frac{\partial U(X, Y)}{\partial X}(X' - X) + \frac{\partial U(X, Y)}{\partial Y}(Y' - Y) > 0.$$

- *if $U(X', Y') \geq U(X, Y)$ then*

$$\frac{\partial U(X, Y)}{\partial X}(X' - X) + \frac{\partial U(X, Y)}{\partial Y}(Y' - X) \geq 0.$$

Proof of Proposition 1.   Suppose that the allocation $(X_1, \ldots, X_n, Y)$ satisfies the Samuelson conditions and suppose that the allocation $(X'_1, \ldots, X_n,' Y')$

is Pareto superior to $(X_1, \ldots, X_n, Y)$. Then for some i, $U_i(X_i', Y') > U_i(X_i, Y)$ and for all i $U_i(X_i', Y') \geq U_i(X_i, Y)$. From Lemma 1 it follows that for all i,

$$\frac{\partial U_i(X_i, Y)}{\partial X_i}(X_i' - X_i) + \frac{\partial U_i(X_i, Y)}{\partial Y}(Y' - Y) \geq 0 \qquad (4.12)$$

with a strict equality for all i such that $U_i(X_i', Y') > U(X_i, Y)$. Since, by assumption, $\partial U_i(X_i, Y)/\partial X_i > 0$, it follows that

$$X_i' - X_i + \frac{\frac{\partial U_i(X_i, Y)}{\partial Y}}{\frac{\partial U_i(X_i, Y)}{\partial X_i}}(Y' - Y) \geq 0 \qquad (4.13)$$

for all i with strict inequality for some i.

Adding these inequalities over all consumers, we find that

$$\sum_{i=1}^{n}(X_i' - X_i) + \sum_{i=1}^{n}\left(\frac{\frac{\partial U_i(X_i, Y)}{\partial Y}}{\frac{\partial U_i(X_i, Y)}{\partial X_i}}\right)(Y' - Y) \geq 0. \qquad (4.14)$$

Using the Samuelson condition as stated in Equation 4.7, this expression simplifies to

$$\sum_{i=1}^{n}(X_i' - X_i) + c(Y' - Y) \geq 0. \qquad (4.15)$$

But Expression 4.15 implies that

$$\sum_{i=1}^{n}X_i' + cY' > \sum_{i=1}^{n}X_i + cY = W. \qquad (4.16)$$

From Expression 4.16, we see that the allocation $(X_1', \ldots, X_n,' Y')$ is not feasible. Therefore there can be no feasible allocation that is Pareto superior to $(X_1, \ldots, X_n, Y)$.

Efficiency and Maximizing Sum of Utilities: Sufficient Doesn't Mean Necessary

The question you see below appeared on the Spring 2011 prelim at UCSB. Many students who otherwise did very well on the exam made a "bad" mistake in answering this problem. The fact that several students were confused suggests that there is something useful to be learned from studying this problem.

The question

Two people share an apartment. Their utilities depend on the cleanliness of the apartment and the hours spent doing housework. Both people are averse to housework. Their utility functions are:

$$U^A(h_A, c) = c - h_A^3$$

for person A and

$$U^B(h_B, c) = c - \frac{1}{3}h_B^3$$

for person B, where c is the level of cleanliness of the apartment and h_i, $i = A, B$ is the number of hours of housework by i. The level of cleanliness is determined by 'production function'

$$C = h_A + h_B.$$

Suppose that both A and B are endowed with a large amount of time which is denoted by H.

a) Write out a generic feasible allocation in terms of hours of housework from each person and the cleanliness for the apartment.

b) Establish the Samuelson condition for interior Pareto optimal allocations.

c) Show that there are Pareto optimal allocations at which A does more housework than B. Are there Pareto optimal allocations requiring more housework from person B? Explain why or why not.

d) Suppose that A and B independently decide how much housework they will do. Find the hours of housework done by each of them in Nash equilibrium, and show that both of them would be better off if each of them could do a little extra housework.

Most people answered Part a correctly. If Person i has a total amount of time T_i available, then the set of feasible allocations consists of combinations (h_A, h_B, c) such that $0 \leq h_A \leq T_A$, $0 \leq h_B \leq T_B$ and $h_A + h_B = c$.

Maximize the sum of utilities?

But surprisingly many screwed up Part b. They asserted that a *necessary* condition for an allocation to be Pareto optimal is that it maximizes $U^A(h_A, c) + U^B(h_B, c)$ over the set of feasible allocations. This is simply not true. It is true that maximizing $U^A(h_A, c) + U^B(h_B, c)$ is a *sufficient* condition, but this is NOT a *necessary* condition. Lets see where this mistake led those who made it.

Suppose that you maximize

$$U^A(h_A, c) + U^B(h_B, c) = c - h_A^3 + c - \frac{1}{3}h_B^3 \qquad (4.17)$$

subject to the constraints that $0 \le h_A \le T_A$, $0 \le h_B \le T_B$ and $h_A + h_B = c$. Since $h_A + h_B = c$, we can rewrite expression 4.17 as

$$2(h_A + h_B) - h_A^3 + c - \frac{1}{3}h_B^3 \qquad (4.18)$$

and find its maximum by setting the derivatives with respect to h_A and h_B equal to zero. When we do this, we find that h_A and h_B are uniquely determined. Setting the two partial derivatives of Expression 4.18 equal to zero, we have

$$2 - 3h_A^2 = 0$$

and

$$2 - h_B^2 = 0,$$

which implies that

$$h_A = \sqrt{\frac{2}{3}}$$

and

$$h_B = \sqrt{2}.$$

This would mean that at every Pareto optimal allocation, Person A does less housework than person B. Can this possibly be right? Must it be that in *every* Pareto optimal allocation, the person who hates housework less will have to do more of it?

This is easily sorted out if you remember the difference between necessary and sufficient. Indeed it is true that maximizing the sum of utilities is a *sufficient* condition for Pareto optimality. (I leave that for you to prove as *Exercise 1.*) But maximizing the sum of utilities is not a necessary condition for Pareto optimality. For example, consider the allocation that maximizes

$U^A(h_A, c) + 4U^B(h_B, c)$. This allocation is also Pareto optimal. (I leave this for you to prove as *Exercise 2*.) Substituting $h_A + h_B$ for c, we have

$$U^A(h_A, c) + 4U^B(h_B, c) = 5(h_A + h_B) - h_A^2 - \frac{5}{3}h_B^2 \qquad (4.19)$$

Taking derivatives you will find that this expression is maximized when $h_A = \sqrt{5/3}$ and $h_B = \sqrt{5/4}$. In this Pareto optimum we see that Person A does more housework than Person B.

More Pareto Optima

In fact, the Pareto optima include all allocations that maximize functions of the form

$$U^A(h_A, h_A + h_B) + KU^B(h_B, h_A + h_B) \qquad (4.20)$$

over the set of feasible h_A and h_B for some $K > 0$. For the utility functions of this problem, we have

$$U^A(h_A, h_A + h_B) + KU^B(h_B, h_A + h_B) = (K+1)(h_A + h_B) - h_A^3 - \frac{K}{3}h_B^3.$$

We can solve explicitly for the maximizing values of h_A and h_B as a function of K. The first order conditions are

$$(K+1) - 3h_A^2 = 0 \quad \text{and} \quad (K+1) - Kh_B^2 = 0$$

which implies that there are Pareto optima whenever

$$h_A = \sqrt{\frac{K+1}{3}} \quad \text{and} \quad h_B = \sqrt{\frac{K+1}{K}}$$

for some $K > 0$. We see that for the Pareto optima in which $K > 3$, person A does more housework than B and for those in which $K < 3$, person B does more housework.

A remark on Necessity and Sufficiency

If you did Exercise 2, you will see that it is true that maximizing $u^A + Ku^B$ for any $K \geq 0$ is sufficient for Pareto optimality.

The question didn't ask you to show this, but it is also true that if u^A and u^B are both concave functions , then a necessary condition for an allocation to be Pareto optimal is that it either maximizes $u^A + Ku^B$ for *some* $K \geq 0$ or simply maximizes u^B. One proof of this proposition is to note that if

these functions are concave, then the "utility possibility set" is a convex set. A Pareto optimum must be on the northeast boundary of this set. Applying the separating hyperplane theorem, we have the desired result.

"Samuelson conditions"

What about the Samuelson conditions? Is there a general statement that we can make about the sum of the marginal rates of substitution between doing housework and having a clean house. This is not quite so obvious, since there is no "money good" that can be exchanged between the two parties. But, yes there is such a condition. Here is one way to find it.

Let us denote partial derivatives of the function U^A with respect to its first and second arguments by U_1^A and U_2^A and those of U^B with respect to its first and second arguments by U_1^B and U_2^B. We seek to maximize the expression in 4.20. Setting partial derivatives equal to zero, we have

$$U_1^A(h_A, h_A + h_B) + U_2^A(h_A, h_A + h_B) + KU_2^B(h_A, h_A + h_B) = 0 \quad (4.21)$$

$$U_2^A(h_A, h_A + h_B) + K\left(U_1^B(h_A, h_A + h_B) + U_2^B(h_A, h_A + h_B)\right) = 0 \quad (4.22)$$

From Equation 4.22 it follows

$$U_2^A(h_A, h_A + h_B) = -K\left(U_1^B(h_A, h_A + h_B) + U_2^B(h_A, h_A + h_B)\right) \quad (4.23)$$

Substitute the expression on the right side of Expression 4.23 for $U_2^A(h_A, h_A + h_B)$ in 4.21 and rearrange terms to find that

$$K = \frac{U_1^A(h_A, h_B)}{U_1^B(h_A, h_B)}.$$

Substitute this expression for K in Equation 4.21. Now you have

$$U_1^A(h_A, h_A + h_B) + U_2^A(h_A, h_A + h_B) + \left(\frac{U_1^A(h_A, h_B)}{U_1^B(h_A, h_B)}\right) U_2^B(h_A, h_A + h_B) = 0$$
$$(4.24)$$

Divide both sides of this equation by $U_1^A(h_A, h_A + h_B)$ and you have

$$1 + \frac{U_2^A(h_A, h_A + h_B)}{U_1^A(h_A, h_A + h_B)} + \frac{U_2^B(h_A, h_A + h_B)}{U_1^B(h_A, h_A + h_B)} = 0$$

which says that the sum of the two persons' marginal rates of substitution between a clean house and working on cleaning the house must be -1 at a Pareto optimum.

Lagrangean Approach

Alternatively, one can set up the problem as a Lagrange multiplier problem in which you note that an allocation in which person 1 does $\bar{h}_A$ and person 2 does $\bar{h}_B$ will be Pareto optimal if and only if it is impossible to find a feasible allocation of labor (h_A, h_B) that is better for Person A than $(\bar{h}_A, \bar{h}_B)$ and no worse for Person B. This will be the case if $(\bar{h}_A, \bar{h}_B)$ maximizes $U^A(h_A, h_A + h_B)$ subject to the constraint that

$$U^B(h_A, h_A + h_B) \geq \bar{U}^B$$

for some choice of $\bar{U}^B$. The Lagrangean for this problem is

$$U^A(h_A, h_A + h_B) - \lambda \left(\bar{U}^B - U^B(h_B, h_A + h_B) \right).$$

Following through with the calculations will give you the same conditions we found previously.

Exercises

4.1 Muskrat, Ontario, has 1,000 people. Citizens of Muskrat consume only one private good, Labatt's ale. There is one public good, the town skating rink. Although they may differ in other respects, inhabitants have the same utility function. This function is $U_i(X_i, Y) = X_i - 100/Y$, where X_i is the number of bottles of Labatt's consumed by citizen i and Y is the size of the town skating rink, measured in square meters. The price of Labatt's ale is $1 per bottle and the price of the skating rink is $10 per square meter. Everyone who lives in Muskrat has an income of $1,000 per year.

a). Write out the equation implied by the Samuelson conditions.

b). Show that this equation uniquely determines the efficient rink size for Muskrat. What is that size?

4.2 Cowflop, Wisconsin, has 1,100 people. Every year they have a fireworks show on the fourth of July. The citizens are interested in only two things – – drinking milk and watching fireworks. Fireworks cost 1 gallon of milk per unit. Everybody in town is named Johnson, so in order to be able to identify each other, the citizens have taken numbers 1 through 1,100. The utility function of citizen i is

$$U_i(x_i, y) = x_i + \frac{a_i \sqrt{y}}{1000}$$

where x_i is the number of gallons of milk per year consumed by citizen i and y is the number of units of fireworks exploded in the town's Fourth of July extravaganza. (Private use of fireworks is outlawed). Although Cowflop is quite unremarkable in most ways, there is one remarkable feature. For each i from 1 to 1,000, Johnson number i has parameter $a_i = i/10$. Johnsons with numbers bigger than 1,000 have $a_i = 0$. For each Johnson in town, Johnson i has income of $10 + i/10$ units of milk.

a). Find the Pareto optimal amount of fireworks for Cowflop.

> **Hint:** *It is true that you have to sum a series of numbers. But this is a series that Karl Friedrich Gauss is said to have solved when he was in second grade.*

4.3 Some miles west of Cowflop, Wisc. is the town of Heifer's Breath. Heifer's Breath, like Cowflop has 1000 people. As in Cowflop, the citizens are interested only in drinking milk and watching fireworks. Fireworks cost 1 gallon of milk per unit. Heifer's Breath has two kinds of people, Larsons and Olsens. The Larsons are numbered 1 through 500 and the Olsen's are numbered 1 through 500. The Larsons have all Cobb-Douglas utility functions $U_L(x_i, y) = x_i^\alpha y^{1-\alpha}$ and the Olsons all have utility functions $U_O(x_i, y) = x_i^\beta y^{1-\beta}$.

a). Write an expression for the optimal amount of public goods as a function of the parameters of the problem.

b). If $\alpha = \beta$, show that the Pareto optimal amount of public goods depends on the aggregate income in the community, but not on how that income is distributed.

4.4 Let $U_C(X_C, Y) = X_C + 2\sqrt{Y}$ and $U_D(X_D, Y) = X_D + \sqrt{Y}$. Suppose that $c = 1$.

a). Determine the amount of Y that must be produced if the output is to be Pareto optimal and if both persons are to have positive consumption of private goods.

b). Find and describe the set of Pareto optimal allocations in which one or the other person consumes no private goods. Show that at these Pareto optima, the Samuelson conditions do not necessarily apply.

c). Draw the utility possibility frontier.

d). Write down an explicit formula for the straight line portion of the utility possibility frontier.

e). Write down explicit expressions for the curved portions of the utility possibility frontier.

4.5 Cecil and Dorothy found a place to live rent-free. Now they are deciding whether to buy a car. They have a total income of $1000 and a car would cost $400. Cecil's utility is given by $X_C(1+Y)$ and Dorothy's by $X_D(3+Y)$ where $Y = 1$ if they buy a car and $Y = 0$ if they do not, and where X_C and X_D are the amounts that Cecil and Dorothy spend on private goods.

1. Write an equation for the utility possibility frontier if they are not allowed to buy a car and another equation for the utility possibility frontier if they must buy a car.

2. Graph these two utility possibility frontiers and shade in the utility possibility set if they are free to decide whether or not to buy a car.

3. Suppose that if they don't purchase the car, Cecil and Dorothy split their money equally. Could they achieve a Pareto improvement by buying the car? Show your answer in two ways (i) Compare the sum of willingnesses to pay to the cost of the car. (ii) Show whether or not the outcome where each has $500 and they have no car produces a point on the overall utility possibility frontier.

4. Suppose that if they don't purchase the car, income would be divided so that $X_C = 650$ and $X_D = 350$. Could they achieve a Pareto improvement by buying the car? Show your answer in two ways (i) Compare the sum of willingnesses to pay to the cost of the car. (ii) Show whether or not the situation where $X_C = \$650$ and $X_D = \$350$ and they have no car is on the overall utility possibility frontier.

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