Subgame-Perfect Nash Equilibrium
Overview of this part of the course:

- Sequential rationality
- Backwards induction
- Subgame-perfect Nash equilibrium
- Oligopoly application
- Bargaining
- Repeated games

After this: Games with incomplete information (Bayesian games)
In-class game: nuclear threat

Ohio and Michigan are constantly on the brink of war. Michigan long ago acquired nuclear weapons as a means to deter attacks from Ohio. Ohio has been furiously trying to develop nuclear weapons itself, but Michigan has said that if Ohio continues its nuclear program, Michigan will bomb Ohio, igniting a deadly and costly regional conflict. How do you predict this scenario will play out? Will Ohio continue to weaponize or will it quit? Will Michigan bomb Ohio or relent on its threat?

Let’s formalize this game to help us analyze it.
Nuclear Threat in normal form

How does your assessment of the payoffs of the participants affect your prediction?

<table>
<thead>
<tr>
<th>Weaponize</th>
<th>Bomb</th>
<th>Relent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>??</td>
<td>??</td>
</tr>
<tr>
<td>Quit</td>
<td>??</td>
<td>??</td>
</tr>
</tbody>
</table>

Compared to the challenge of *faithfully* formalizing the structure and payoffs of the game, applying solution concepts seems pretty straightforward.
Nuclear Threat in extensive form

Timing should matter, and normal form ignores it. Let’s use the extensive form.

Ohio can’t really make it’s decision until it considers how Michigan will react → first look at the bottom of the game tree.
Sequential rationality

Suppose that Nuclear Threat players’ incentives are just like those in the Entry Game

Recall: (W,R) and (Q,B) both NE, but you didn’t like (Q,B)
What’s wrong with (Q,B)?
Threatening to bomb is rational *ex ante*, given Q, but it wouldn’t be rational if Michigan were ever called upon to actually act
Sequential rationality

- Bombing (in this case) is not *Sequentially rational*

- It doesn’t make sense to predict actions that are not sequentially rational

- We can eliminate actions that are not sequentially rational, node by node, starting from the bottom of the game tree

- This eliminates threats that are not credible

- This is called *Backwards Induction*

Strategies profiles that survive backwards induction must be Nash equilibria... but not all Nash equilibria survive backwards induction.
Backwards Induction

Try it on this game:

```
  1
 / \   / \   / \  
2  L  2  R  2  
/ \  / \  / \  / \  
1  a  2,1  b  0,3  0  3  c  2,2  1  1,4
  \  \  \  \  \  
  4,0  0,3  2,2  1,4
```

List the strategies that survive backwards induction
Backwards Induction

What happens if there are ties among the payoffs?

Backwards induction does not necessarily yield a unique prediction
Subgame Perfection

- Applying backwards induction to these games is kind of like applying the concept of dominance—eliminating conditionally dominated strategies—taking into account *sequential* rationality

- Is there a refinement of Nash equilibrium that incorporates *sequential* rationality?

- Yes, many. We call this concept “perfect”

Next solution concept: subgame-perfect Nash equilibrium (SPNE)
Subgames & Subgame Perfection

- Subgame: a part of the game tree that itself constitutes a well-defined game tree
- Formally: a decision node initiates a *subgame* if neither it nor any of its successors are in an information set that contains nodes that are not successors to it.
- Note: a game is always a subgame of itself
- Note: make sure you examine may games to gain familiarity with this concept

Subgame-perfect Nash equilibrium: a strategy profile that specifies a Nash equilibrium in *every* subgame
SPNE & Backwards Induction

Let’s practice BI in this game:

```
<table>
<thead>
<tr>
<th></th>
<th>Out</th>
<th>In</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>3, 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

What is the BI strategy profile?

**Figure:** Sequential BoS anyone?
SPNE & Backwards Induction

Now find SPNE:

![Game Tree Diagram]

**Figure:** Sequential BoS anyone?

What is the SPNE?
Are SPNE and BI identical? NO!

Figure: BoS anyone?
SPNE & Backwards Induction

- BI and SPNE make identical predictions in games with perfect information, but...
- Dominance has no bite in BoS
- BI makes no prediction about behavior in the BoS subgame

![Game Tree Diagram](Figure: BoS anyone?)
SPNE & Backwards Induction

- Hard to apply BI to games with imperfect information (non-singleton information sets)
- But, what if the subgame were PD?

If subgame with imperfect info is dominance solvable, you can use BI
SPNE & Backwards Induction

- NE does make predictions in BoS, namely 2 PSNE (ignore mixed-strategy)
- SPNE eliminates any NE in the overall game that specify non-NE strategies in BoS subgame

Figure: BoS anyone?
Zermelo’s Theorem

Relates to BI: In finite, two-player, “win-lose-draw” games of perfect information, either

- one player has a strategy that guarantees a win, or
- both players have a strategy that guarantees a draw

Applied to chess this means...

chess is solvable by BI!