Repeated Games
Warm up: bargaining

Suppose you and your Qatz.com partner have a falling-out. You agree set up two meetings to negotiate a way to split the value of your assets, which amount to $1 million if liquidated without delay. The first meeting will involve just the two of you. You show up with a proposal, which your partner can either accept or reject. If she rejects the proposal, you will each hire a team of lawyers to help negotiate the dissolution of your partnership and have to pay a 40% fee to your lawyers. At the second meeting you again propose a split, which your partner can accept or reject. If she rejects at the second meeting, it will take a full year to liquidate your assets (qat fotoz), which is enough time for their value to drop to just $400000, which will be split evenly between the two of you—before you each pay your lawyers their 40% fee.

Describe the SPNE of this game
Warm up: repeated games

Election dilemma: Both political parties just spent $1 billion (each) on the presidential race. If only one side spent the money, it would have the advantage. If neither spends, they are evenly matched, but the world has an extra $2 billion to spend on more productive things.

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<thead>
<tr>
<th></th>
<th>Don’t</th>
<th>Spend</th>
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<tr>
<td>Don’t</td>
<td>2, 2</td>
<td>0, 3</td>
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<tr>
<td>Spend</td>
<td>3, 0</td>
<td>1, 1</td>
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We play this game every 4 years

Can a more cooperative outcome be sustained in the long term?
How?
Repeated Games

Overview

- Reputation and long term relationships matter
- We face many strategic situations repeatedly
- We can think of those repeated situations as one big game of repeated stages
- It’s always a SPNE of the repeated game to play a sequence of NE from the stage game, but...
- If players are sufficiently patient, they can sustain more cooperative outcomes in the long run, even outcomes that are not NE
A game

Let’s play this game

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
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<tbody>
<tr>
<td>A</td>
<td>4,3</td>
<td>0,0</td>
<td>1,4</td>
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<tr>
<td>B</td>
<td>0,0</td>
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A finitely repeated game

Let’s play twice in a row!

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A finitely repeated game

Suppose this game is repeated twice ($T = 2$) and players care about the sum of the payoffs from the two stages.

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**Figure:** The stage game from Watson (page 259).

Can players sustain cooperation?
First, what does cooperation mean in this game? What are NE?

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**Figure:** The stage game from Watson (page 259).

**NE:** \((B, Y)\) and \((A, Z)\).
A finitely repeated game

Is either NE efficient?

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Figure: The stage game from Watson (page 259).

(B, Y) is inefficient compared to (A, X)

So what do subgame-perfect Nash equilibria of the repeated game look like? Is there any way that the players can cooperate and sustain (A, X) in equilibrium?
A finitely repeated game

Stage game vs. the repeated game

- Stage game = the game played in each period of the repeated game
- We know strategies in stage game
- What do strategies in the repeated game look like?
- Behavior can be conditioned on *entire* history of relationship
- What does extensive form look like?
- Get’s complicated to draw, describe, so we think about subgames
A finitely repeated game

Player 1 can earn at most $2 + 2 = 4$ if on NE are played in equilibrium. How can she improve on this? If $(A, X)$ is played then she is much better off, and Player 2 is better off compared to at least one of the stage-game NE. To sustain $(A, X)$ in the first stage, 2 has to threaten punishment in the second stage for failure to comply. What’s the worst payoff that 1 can give 2 in the stage game? The lowest payoff for 2 is zero, but 1 can’t guarantee that payoff for Player 2. If 1 chooses $A$, 2 could best respond with $Z$, if 1 chooses $B$, then 2 could best respond with $Y$. So the most damage that 1 can do to 2 is to play $B$, yielding a payoff of 1. We call the worst payoff that 1 can force on player 2, player 2’s minmax payoff.

Definition

Player $i$’s minmax payoff is the solution to the problem

$$\min_{s_{-i} \in S_{-i}} \left( \max_{s_i \in S_i} u_i(s_i, s_{-i}) \right).$$
Sustaining cooperation

- Worst punishment that 1 can threaten is to choose $B$
- We require threats to be credible when we use the concept of subgame perfection.
- To be SPNE, the players have to be playing a NE in the each subgame that can occur in the second stage.

So Player 1’s has to look something like this:

- If you play nicely, i.e. you do your part to play $(A, X)$ in the first stage, then you will be rewarded with the good NE (for you) $(A, Z)$ in the second stage.
- If you don’t play nicely, I will punish you by playing $B$ and we will be stuck in the $(B, Y)$ NE in the second stage.

What is 2’s best response to this? Verify SPNE: is 1’s strategy a BR to 2’s BR?
Sustaining cooperation

A separate punishment NE and reward NE in stage 2 make it possible to cooperate on non-NE \((A, X)\) in stage 1.

Can we do something similar to sustain cooperation in PD? **NO!**

- PD has a unique, inefficient NE
- SPNE requires a NE to be played in the final stage
- No way to choose a punishment NE vs. reward NE
- Player 1 has no threats
- Cooperation unravels in *any* finitely-played PD

Is there *any* way to sustain cooperation in a repeated PD?
Infinitely repeated PD

Let’s play!

\[
\begin{array}{cc}
C & D \\
C & 2,2 & 0,3 \\
D & 3,0 & 1,1 \\
\end{array}
\]

Prisoners’ Dilemma

Discount factor: \( \delta = 0.75 \) (Implemented by probability of termination)

Payment = full stream of payoffs
Infinitely repeated PD

- Cooperation unravels in finitely repeated PD because subgame perfection requires \((D,D)\) in final stage.
- What if there is no final stage?
- Discount stage-game payoffs by \(\delta\).
- Low \(\delta\) → players don’t care about future, no cooperation.
- High \(\delta\) → future, reputation matter so cooperation possible.

How high must \(\delta\) be to sustain cooperation?
Infinitely repeated PD

- We rely on a “grim-trigger” strategy:
  - Cooperate as long as other player has been cooperating
  - If opponent defects, play D forever

- Can sustain cooperation if discounted stream of (C,C) payoffs is no less than stream of payoffs following D

How do we calculate the value of an infinite stream of payoffs?
Calculating discounted payoffs

We apply a discount factor $\delta$ to the stream of payoffs: if you get a payoff of 1 each period, your discounted payoff is

$$s = 1 + \delta + \delta^2 + \delta^3 + \cdots,$$

which we can simplify by noting that $s = 1 + \delta s$, and solving to get

$$s = \frac{1}{1 - \delta}.$$

So if your payoff is $x$ in every period, your discounted payoff is $\frac{x}{1 - \delta}$. 
Cooperation in infinitely repeated PD

\[
\begin{array}{c|cc}
\text{ } & C & D \\
\hline
C & 2,2 & 0,3 \\
D & 3,0 & 1,1 \\
\end{array}
\]

Prisoners’ Dilemma

- Value of cooperating is \( \frac{2}{1-\delta} \)
- Value of defecting is \( 3 + \delta \frac{1}{1-\delta} \) (3 once, then 1 forever)
- No incentive to deviate when \( \delta \geq 1/2 \)

If players are sufficiently patient, they can sustain cooperation
Cooperation in infinitely repeated PD

What about these other versions of the PD (from PS2)

(a)

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(b)

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<td>S</td>
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<td>1,2</td>
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(c)

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<tr>
<td>A</td>
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Cooperation in infinitely repeated PD

a) Greater incentive to deviate requires greater patience

b) Different players face different incentives. Cooperation requires enough patience for both players

c) One player needs no punishment threat
Cool Down: BoS variant

Player 1 is unsure whether Player 2 wants to go out with her or avoid her, and thinks that these two possibilities are equally likely. Player 2 knows Player 1’s preferences. So Player 1 thinks that with probability $1/2$ she is playing the game on the left and with probability $1/2$ she is playing the game on the right.

Player 2 wishes to meet:

\[
\begin{array}{cc}
O & B \\
O & 2, 1 & 0, 0 \\
B & 0, 0 & 1, 2 \\
\end{array}
\]

Player 2 wishes to avoid:

\[
\begin{array}{cc}
O & B \\
O & 2, 0 & 0, 2 \\
B & 0, 1 & 1, 0 \\
\end{array}
\]

2 wishes to meet

2 wishes to avoid