When Non-transitive Relations have Maximal Elements and Competitive Equilibrium Can't be Beat

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In 1967, I had the great good fortune of being a new assistant professor at Washington University when Trout Rader was a young associate professor with no other theorists around him to talk to. For me these were exciting times. Here was a person with powerful technical skills, full of imaginative ideas, and eager to share them–someone to learn from, to admire, and to emulate. Trout was working on a two-volume treatise on economic theory, and I would read the chapters as he produced them. Trout's first drafts were not paragons of clarity, but they were full of clever stuff and I could get him to try to explain this stuff to me. At least at first, the main coin I could offer in return for his tutelage was typographical errors from his manuscript. Luckily for me, this currency was readily available and could always be redeemed for good value. Trout showed me that mathematical economics led to depths of reasoning and logical subtlety that I had not dared to hope existed in economics.

Trout wrote his Ph. D. dissertation on competitive equilibrium theory, a subject that I had almost entirely evaded in graduate school. Trout convinced me that this was a rich subject whose applications were only beginning to be tapped. With his encouragement, I wrote papers on general equilibrium in a slave economy, general equilibrium with benevolent consumers, and general equilibrium with public goods. As I worked on these questions, I found that competitive equilibrium as described in Debreu's *Theory of Value* was not as general as it could be or as one would like it to be in "realistic" applications.

I accumulated a stock of ideas and tricks to use in existence proofs, many of which were inspired by Debreu's 1962 paper, McKenzie's 1959 and 1961 papers, the drafts of Rader's 1972 book on general equilibrium, and Sonnenschein's (1971) paper on non-transitive consumers. In 1973, I put the tricks that I knew into a paper called "Competitive Equilibrium Without Transitivity, Local Nonsatiation or Monotonicity". Not only were the consumers in that paper locally sated and nonmonotonic, they were afflicted with Sonnenscheinian nontransitivity.[1] Not long after the 1973 paper was written, Trout and I came across Gale and Mas Collel's brilliant (1975) reformulation of equilibrium existence theory, which handled nontransitive consumers much more elegantly and powerfully. We collaborated with Robert Parks on an exploration of Gale and Mas Collel's open graph assumption (1976). Trout wrote a paper extending his own earlier work on induced preferences and household production to the Gale-Mas Collel environment (1975). I wrote a paper, "The Existence of Maximal Elements and Equilibria in the Absence of Transitivity", (1975) which I believed to be an elegant and useful extension of the Gale Mas Collel theory. The referees of the journal I sent it to did not share my enthusiasm for this paper and I put it in my desk to ripen.

[1] Trout suggested that the paper should be called "On the Existence of Equilibrium in a Lunatic Asylum".

In the meantime, several good papers extended the the Fan, Sonnenschein and Gale-Mas Collel theorems. Papers dealing with finite-dimensional spaces include Shafer and Sonnenschein (1975), (1976), Shafer (1976), Borglin and Keiting (1976), Gale and Mas Collel (1977), McKenzie (1981), Geistdoerfer-Florenzano (1982), and Schofield (1984), Blume (1986). Other writers worked on extending these results to infinite-dimensional commodity spaces and to infinite numbers of consumers. The latter include Aliprantis and Brown (1983), Florenzano (1983), Yannelis and Prabhakar (1983), Khan (1984), Toussaint (1984) and Mehta (1987).

Most of the theorems found here are taken from my unpublished 1975 paper. Almost all of these results can be found scattered somewhere in the literature, but an expository paper laying them out in a consistent format seems called for. The results on existence of competitive equilibrium fall into the category that McKenzie (1981) calls the "modern form of the classical theorem on existence of competitive equilibrium".[2] I have added some new results stimulated by the work of Geistdoerfer-Florenzano (1982), Moore (1981), McKenzie (1981) and Mehta (1987) as well as several examples, motivating discussions and references to more recent literature.

This paper shows that Nash equilibrium and competitive equilibrium exist as maximal elements of judiciously chosen non-transitive binary relations that are continuous enough and convex enough to have maximal elements. That this should be possible is strongly suggested by the structure of Gale and Mas Collel's proofs. A difficulty with directly adopting the Ky Fan theorem to this purpose is that the most natural candidates for such a binary relation lack the open graph property. In this paper, the convexity and continuity assumptions made by Fan, Sonnenschein, and Gale-Mas Collel are relaxed in such a way as to unify and extend their results.

1. A Generalization of Theorems by Fan and Sonnenschein

In 1961, Ky Fan discovered the remarkable fact that even a nontransitive binary relation must take maximal elements on compact, convex sets so long as there is sufficient continuity and convexity. This idea was introduced to economists by Hugo Sonnenschein (1971) who independently proved a similar result and showed that an interesting demand theory could be constructed without transitivity.

Let S be a set and $P \subset S \times S$ a binary relation. If $(y, x) \in P$, we write yPx. Where $x \in S$, let the "better than" sets be denoted by $P(x) = \{y|yPx\}$ and the "worse than" sets by $P^{-1}(x) = \{y|xPy\}$. Where $X \subset S$, $x^* \in X$ is a *maximal element* for P on X if $P(x^*) \cap X = \emptyset$.

Fan (1961) and Sonnenschein (1971) have proved the following results on the existence of maximal elements.

[2] McKenzie defines the "classical theory" as dealing with finite numbers of consumers and a finite dimensional commodity space. Although many of our results on existence of maximal elements apply in infinite-dimensional spaces, the competitive equilibrium theory here is all finite-dimensional.

Ky Fan's Theorem. *Let $P \subset S \times S$ be a binary relation where S is a linear topological space and where*

(i) P is irreflexive,

(ii) For all $x \in S$, $P(x)$ is convex or empty,

(iii) (Open graph.) P is an open set in $S \times S$.

Then if $X \subset S$ is non-empty, convex, and compact, there exists a maximal element for P on S.

Hugo Sonnenschein's Theorem. *Let $P \subset \Re^n \times \Re^n$ be a binary relation such that:*

(i) P is asymmetric,

(ii) For all $x \in \Re^n$, $P(x)$ is convex or empty,

(iii) (Open "worse than" sets.) For all $x \in \Re^n$, $P^{-1}(x)$ is open.[3]

Then if $X \subset \Re^n$ is non-empty, convex and compact, there exists a maximal element x^* for P on X.

Neither of these two theorems implies the other. Fan's open-graph assumption is stronger than Sonnenschein's assumption of open "worse than" sets. On the other hand, Fan's theorem applies to arbitrary linear topological spaces, while Sonnenschein's theorem is proved only for finite-dimensional Euclidean space.

Theorem 1 generalizes both Fan's theorem and Sonnenschein's theorem.

Theorem 1. *Let $P \subset S \times S$ be a binary relation where S is a linear topological space and where:*

1.A) For all $x \in S$, $x \notin \mathrm{con}\,\mathrm{P(x)}$.[4]

1.B) For all $x \in S$, $P^{-1}(x)$ is open.

Then if $X \subset S$ is non-empty, convex, and compact, there exists a maximal element for P on X.

To prove Theorem 1, we use a lemma due to Ky Fan, which extends the well-known theorem of Knaster, Kuratowski, and Mazurkiewicz to topological linear spaces of arbitrary dimension.

Lemma 1. *(Ky Fan) Let S be a linear topological space and $X \subset S$. For each $x \in X$, let $F(x)$ be a closed set in S such that:*

(i) The convex hull of any finite subset $\{x_1, ..., x_n\}$ of X is contained in $\bigcup_{i=1}^{m} F(x_i)$.

(ii) $F(x)$ is compact for at least one $x \in X$.

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

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3 Bergstrom, Parks and Rader (1976) call this property "open lower sections", Schofield (1983) calls it "lower demi-continuity".

4 For any set, S, $\mathrm{con}\,\mathrm{S}$ denotes the smallest convex set containing S. Schofield defines a relation with property 1.A) to be "semi-convex".

Proof of Theorem 1:

Let $\{x_1, ..., x_n\}$ be a finite subset of X. If $z \in \bigcap_{i=1}^{n} P^{-1}(x_i)$, then by Assumption (i) of Theorem 1, $z \notin \mathrm{con}\{\mathrm{x_1, ..., x_n}\}$. Therefore $\bigcap_{i=1}^{n} P^{-1}(x_i) \subset (\mathrm{con}\{\mathrm{x_1, ..., x_n}\})^{\mathrm{c}}$ and hence $\mathrm{con}\{\mathrm{x_1, ..., x_n}\} \subset \left(\bigcap_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{P^{-1}(x_i)}\right)^{\mathrm{c}} = \bigcup_{\mathrm{i=1}}^{\mathrm{n}} \left(\mathrm{P^{-1}(x_i)}\right)^{\mathrm{c}}$. For all $x \in X$, let $F(x) = (P^{-1}(x))^c \cap X$. Then the sets $F(x)$ satisfy condition (i) of Lemma 1. Also, since $P^{-1}(x)$ is open (by Assumption 1.B) and X is compact, $F(x)$ is compact for all $x \in X$. Therefore according to Lemma 1, $\bigcap_{x \in X} F(x) \neq \emptyset$. Therefore $x^* \in \bigcap_{x \in X} F(x)$ for some x^*. From the definitions, it follows that x^* is a maximal element for P on X. ∎

Remarks on the convexity assumption

Fan assumes that $x \notin P(x)$ (P is irreflexive). Sonnenschein assumes that P is asymmetric, which implies that P is irreflexive. Both Fan and Sonnenschein assume that $P(x)$ is either convex or empty, and hence that $\mathrm{con\,P(x) = P(x)}$. Therefore both authors' assumptions imply Theorem 1's assumption that $x \notin \mathrm{con\,P(x)}$.

There are economically interesting examples of preferences that satisfy $x \notin \mathrm{con\,P(x)}$, but do not have $P(x)$ convex. Consider a consumer who consumes two desirable goods, but finds that fractional units of both goods are useless. For this consumer, $(2, 1) \in P(1, 1)$ and $(1, 2) \in P(1, 1)$, but $.5(2, 1) + .5(1, 2) = (1.5, 1.5) \notin P(1, 1)$. Although this example violates convexity of $P(x_1, x_2)$, it does not violate $x \notin P(x)$.

For another example, consider a consumer who takes the trouble to recognize that commodity bundle x differs from y only if the Euclidean distance $d(x, y)$ between these two bundles exceeds some threshold amount ϵ. Let the consumer also have a continuous strictly quasi-concave utility function, u_i and suppose that she prefers x to y if and only if $u_i(x) > u_i(y)$ and $d(x, y) > \epsilon$. This consumer would not have convex preferred sets, but her preferences do have the property that $x \notin \mathrm{con\,P(x)}$.

Remarks on the continuity assumption

Continuous preferences, as defined by Debreu (1959) have open "better than" sets, $P(x)$, and open "worse than" sets, $P^{-1}(x)$. The open-graph assumption made by Ky Fan and also by Gale and Mas Collel is stronger than Debreu's continuity. Bergstrom, Rader and Parks show examples of preferences that do not have open graph but are continuous in the sense of Debreu. They show that Debreu's continuity assumption implies open graph if P is transitive and "order dense". They also adapt a theorem of Shafer (1974) to show that if X is the entire non-negative orthant in $\Re^n$ and if $P(x)$ is a convex set for all $x \in X$, then Debreu's continuity assumption implies the open-graph property.

Theorem 1, like Sonnenschein's theorem, requires only that the "worse than" sets be open. If some commodities are useful only in discrete units, it is reasonable to assume that all "worse than" sets are open, but it is not reasonable to assume that all "better than" sets are open. For example, consider the relation P on $\Re^2$ such that $(x, y)P(x', y')$ if and only if $x + \lfloor y \rfloor > x' + \lfloor y' \rfloor$, where $\lfloor y \rfloor$ is the greatest integer smaller than y. The set $P(1, 1)$, is not open. To see this, notice that $(1, 2)P(1, 1)$, but every open neighborhood of $(1, 2)$ contains points (x, y) such that $(1, 1)P(x, y)$. On the other hand, it is easy to verify that for every point (x, y) in the nonnegative orthant, the set $P^{-1}(x, y)$ is open.

2. Lower Semi-continuous Preferences and Existence of Maximal Elements

Some familiar and useful asymmetric binary relations have neither open "better than" sets nor open "worse than" sets, but are "continuous enough" to take maximal elements on closed, bounded convex sets. Consider the following examples:

- (the lexicographic ordering, P_{lex}) Define P_{lex} on $\Re^2$ so that $(x_1, x_2)P_{lex}(y_1, y_2)$ if $x_1 > y_1$ or if $x_1 = y_1$ and $x_2 > y_2$.
- (the "Nash improvement relation", $\hat{P}$) Consider a game where players have strategy sets X_i and continuous individual preferences P_i defined on the outcome space, $X = \prod_i X_i$. Define $\hat{P}_i$ on X so that $x\hat{P}_i y$ if player i prefers x to y and if x differs from y only by i's choice of strategy. Define $\hat{P}$ on X so that $x\hat{P}y$ if and only if $x\hat{P}_i y$ for *some* player i.

Each of these relations has the property that if xPy then for all points y' "close to y", there are points x', "arbitrarily close to x", such that $x'Py'$. This property is known as *lower semi-continuity* of the correspondence that maps each point x to the set $P(x)$. We will define a relation with this continuity property to be *lower semi-continuous*. In finite-dimensional spaces, we can replace Theorem 1's assumption of open "worse than" sets by the weaker assumption that P is lower semi-continuous.[5] Lower semi-continuity is a nice property to work with because complicated lower semi-continuous binary relations can be built from simple ones. For example, unions, intersections, Cartesian products, projections, and convex hulls of lower semi-continuous correspondences are lower semi-continuous. Several propositions that are useful for identifying and constructing lower semi-continuous correspondences are presented in the Appendix.

Definition 1. *(Lower semi-continuous correspondence.) Let X and Y be topological spaces and 2^Y the set of all subsets of Y (including the empty set). The correspondence $\phi : X \to 2^Y$ is lower semi-continuous (l. s. c.) if for every set V which is open in Y, the set $\{x \in X \mid \phi(x) \cap V \neq \emptyset\}$ is open in X.*

Definition 2. *(Lower semi-continuous binary relation.) Let S be a topological space, $P \subset S \times S$, and $X \subset S$. The relation P is lower semi-continuous on X if the correspondence $\phi : X \to 2^Y$ where $\phi(x) = P(x) \cap X$ is l. s. c. (where X is endowed with the relative topology).*

Theorem 2. *Let $P \subset \Re^n \times \Re^n$ be a binary relation and let $X \subset \Re^n$ be non-empty, convex and compact where:*

[5] This possibility was also noted by Gale and Mas Collel (1979).

2.A) For all $x \in X$, $x \notin \mathrm{con\,P(x)}$.

2.B) P is lower semi-continuous on X.

Then there exists a maximal element for P on X.

To prove theorem 2, we use a lemma which is a special case of Michael's selection theorem (Michael (1956), Theorem 3.1‴).

Lemma 2. *Let $X \subset \Re^n$ and $\phi : X \to 2^X$ a l. s. c. correspondence such that for all $x \in X$, $\phi(x)$ is a non-empty convex set. Then there exists a continuous function $f : X \to X$ such that for all $x \in X$, $f(x) \in \phi(x)$.*

Proof of Theorem 2:

Suppose that for all $x \in X$, $P(x) \cap X \neq \emptyset$. By Assumption 2.B, the correspondence $\phi : X \to 2^X$ where $\phi(x) = P(x) \cap X$ is l. s. c. Then the correspondence $\Psi : X \to 2^X$ where $\Psi(x) = \mathrm{con}\,\phi(\mathrm{x})$ for all $x \in X$ is also l. s. c. (Proposition 9, the Appendix) and $\Psi(x)$ is non-empty and convex for all $x \in X$. By Lemma 2, there exists a continuous function $f : X \to X$ such that for all $x \in X$, $f(x) \in \Psi(x)$. Since X is non-empty, convex, and compact, by Brouwer's fixed point theorem there exists $\bar{x} \in X$ such that $\bar{x} = f(\bar{x}) \in \Psi(\bar{x})$. But then $\bar{x} \in \mathrm{con\,P(\bar{x})}$ which is contrary to Assumption 2.B. Therefore it must be that $P(x^*) \cap X = \emptyset$ for some $x^* \in X$. ∎

As we show in Proposition 2 of the Appendix, P will be l. s. c. if $P^{-1}(x)$ is open for all x. Thus for finite dimensional spaces, Theorem 2 generalizes Theorem 1.

For proving the existence of competitive equilibrium where the set of feasible trades is not necessarily convex, it will be useful to have the following corollary to Theorem 2.

Corollary. *Let P be a binary relation defined on $X \times X$ where $X \subset \Re^n$ and let S be a convex, compact subset of X such that if $x \in S$ and $y \in X \sim S$, then $(x, y) \notin P$. Then if P satisfies conditions 2.A and 2.B of Theorem 2, there exists $x^* \in S$ such that x^* is a maximal element for P on X.*

To prove the Corollary, notice that Theorem 2 implies that P has a maximal element on S and under the assumptions of the corollary, a maximal element of P on S is also a maximal element of P on X.

Extensions to infinite-dimensional spaces

Our Theorem 1 applies in spaces of infinite as well as finite dimension, but requires that "worse than sets" all be open. As Aliprantis and Brown (1983) show, this theorem can be used to establish the existence of competitive equilibrium in an economy with an infinite-dimensional commodity space. Theorem 2 allows us to weaken the assumption of open "worse than" sets to the assumption that preferences are lower semi-continuous, but at the cost of requiring the consumption space is finite-dimensional.

Mehta (1987) proves an alternative weakening of the continuity assumption that allows the existence of maximal elements on infinite-dimensional spaces.[6]

[6] In a separate theorem, Mehta shows that Theorem 2 can be extended to any Banach space if Assump-

Theorem 1′. *(Mehta) Theorem 1 remains true if Assumption 1.A) is replaced by the (weaker) assumption:*
1′.A) if $P(x) \neq \emptyset$, then there exists at least one point in the interior of $P^{-1}(x)$.[7]

Theorem 1′ is stronger than Theorem 1, but not strictly comparable with Theorem 2 since the assumption of weak lower demi-continuity is neither weaker nor stronger than the assumption of lower semi-continuity.

Another infinite-dimensional alternative to Theorem 2 is the the following result:

Theorem 2′. *Let $P \subset S \times S$ where S is a separable Banach space and let $X \subset \Re^n$ be non-empty, convex and compact where:*

2.A) For all $x \in X$, $x \notin conP(x)$.

2.B) P is lower semi-continuous on X.

2′.C) If $P(x) \neq \emptyset$, then either $P(x)$ has an interior point or $P(x)$ is closed.

Proof of Theorem 2′:

We appeal again to Michael's Theorem 3.1‴, which informs us that a lower semi-continuous correspondence has a continuous selection if the image sets are nonempty and convex and either finite-dimensional, closed, or have interior points. If $\mathrm{con\,P(x)} \neq \emptyset$ for all x in then there exists a continuous function f mapping X into itself. By Schauder's fixed point theorem, f must have a fixed point $\bar{x}$. But this can not be since then we would have $\bar{x} \in \mathrm{con\,P(\bar{x})}$, contrary to 2.A). ∎

Other Extensions

It is known that an acyclic binary relation with open "worse than" sets takes maximal elements on compact sets. (Bergstrom (1975), Walker(1977), Birchenhall (1977). Schofield (1984) was the first to relate these results to the Sonnenschein-Fan theorems. Schofield shows that assuming that preferences satisfy a "local" convexity property is equivalent to the nonexistence of "local" cycles. He also shows that if the set X is either a convex set or a smooth manifold of a certain topological type, then the global condition $x \notin \mathrm{con\,P(x)}$ can be replaced by a local version of the same assumption in theorems guaranteeing that P has a maximal element on X.

Mehta (1987) shows that the assumption that the set X is compact can be weakened in interesting ways.

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tion 2.A is strengthened to the assumption that $P(x)$ is closed and convex for all x and $x \notin P(x)$. Of course the assumption that P would not be sustainable in the usual model of strict preference.

[7] Mehta calls this property "weak lower demi-continuity".

3. The Existence of Nash Equilibrium

Nash (1950) showed how to prove the existence of "Nash" equilibrium in a game where the strategy sets are compact and convex and where each individual's preferences are represented by a continuous function that is quasi-concave in his own strategy. Nash observed that with his assumptions, the reaction correspondence (he called it the countering correspondence) has a closed graph and convex image sets and he applied Kakutani's fixed point theorem to show the existence of a fixed point, which in turn must be an equilibrium.

Nash's convexity and continuity assumptions can be relaxed and the transitivity assumption (made implicitly when preferences on outcomes are represented by real-valued payoff functions) can be eliminated. This is done by finding a binary relation for which a maximal element must be a Nash equilibrium and which satisfies the conditions of Theorem 2.

Definition 3. *(Game.) A game $(P_i, X_i)_{i \in I}$ consists of a set of players $I = \{1, ..., m\}$, a strategy set X_i for each $i \in I$, a set of possible outcomes $X = \prod_{i \in I} X_i$, and for each $i \in I$, a preference relation $P_i \subset X \times X$.*

A Nash equilibrium is an outcome $\bar{x} \in X$ such that no player can gain by choosing an alternative strategy if all other players persist in their choice of the strategies named in $\bar{x}$. To establish Nash equilibrium as a maximal element of a binary relation, it is useful to define $\hat{P}_i \subset P_i$ so that $x\hat{P}_i y$ if player i prefers x to y and if player i can single-handedly change the outcome from y to x. Also, define $\hat{P}$ so that $x\hat{P}y$ if *someone* who prefers x to y can single-handedly change the outcome from y to x. An outcome is a Nash equilibrium if and only if it is a maximal element of $\hat{P}$ on the set of possible outcomes.

Definition 4. *(Nash improvement relations.) The Nash improvement $\hat{P}_i \subset X \times X$ for player i is defined so that $x\hat{P}_i y$ if and only if $xP_i y$ and for all $j \neq i$, $x_j = y_j$. The (overall) Nash improvement improvement relation is defined to be $\hat{P} = \bigcup_I \hat{P}_i$. For all $x \in X$, define $P_i^*(x) \subset X_i$ to be the projection of $\hat{P}_i(x)$ on X_i.*

Definition 5. *(Nash equilibrium.) A Nash equilibrium for the game $(P_i, X_i)_{i \in I}$ is a point $\bar{x} \in X$ such that $\hat{P}(\bar{x}) = \emptyset$.*

Theorem 3. *Let $(P_i, X_i)_{i \in I}$ be a game such that $X_i \subset \Re^n$ is non-empty, convex, and compact for all $i \in I$ and:*

3.A) For all $x \in X$ and all $i \in I$, $x \notin \mathrm{con}\,\hat{\mathrm{P}}_\mathrm{i}(\mathrm{x})$.

3.B) For all $i \in I$, $\hat{P}_i$ is lower semi-continuous.

Then there exists a Nash equilibrium.

The continuity assumption in Theorem 3 is that $\hat{P}_i$ is l. s. c. for all $i \in I$. This assumption is weaker than the assumption that P_i is lower semi-continuous and is equivalent to lower semi-continuity of the mapping $P_i^* : X \to X_i$. [8]

[8] Treat P_i^* and $\hat{P}_i$ as correspondences which map x to $P_i^*(x)$ and $\hat{P}_i(x)$, respectively. Since P_i^* is the ith

Proof of Theorem 3:

We will show that $\hat{P}$ satisfies the conditions of Theorem 2. First we show that for all $x \in X$, $x \notin \text{con}\,\text{P}(\text{x})$. If $y \in \text{con}\,\hat{\text{P}}(\text{x})$, then $y = \sum_{j \in J} \lambda_j y^j$ where $J \subset I$, where $\sum_{j \in J} \lambda_j = 1$ and for all $j \in J$, $\lambda_j \geq 0$ and $y^j \in \hat{P}_j(x)$. Then $y - x = \sum_{j \in J} \lambda_j z^j$ where $z^j = y^j - x$. Assumption 3.A requires that $z^j \neq 0$ for $j \in J$. The correspondences $\hat{P}_j$ are constructed so that the z^j's are linearly independent and therefore $y - x \neq 0$. It follows that $x \notin \text{con}\,\hat{\text{P}}(\text{x})$.

Since $\hat{P}_i$ is assumed to be l. s. c. and since unions of l. s. c. correspondences are also l. s. c., (Appendix, Proposition 3), it must be that $\hat{P}$ is l. s. c. It follows from Theorem 2, that $\hat{P}(\bar{x}) = \emptyset$ for some $\bar{x} \in X$. By definition, $\bar{x}$ must be a Nash equilibrium. ∎

4. Existence of Free-Disposal Competitive Equilibrium

In a "Robinson Crusoe" economy,[9] where Robinson has convex preferences and a convex production possibility set, a competitive equilibrium allocation can be found simply by maximizing Robinson's utility on the production possibility set. In general for an exchange economy, even where individual preferences are continuous, transitive, and convex, there is no continuous "aggregate utility function" that can be maximized to find a competitive equilibrium. Nevertheless, under very general conditions there exists a non-transitive binary relation satisfying the conditions of Theorem 2, which is maximized at a competitive equilibrium.

This section proves existence of equilibrium for an exchange economy with free disposal and where consumer preferences are locally nonsatiated. As will be seen later, these assumptions are not essential for existence of equilibrium. Free-disposal and local nonsatiation assumptions can be eliminated, but in order to see the structure of the existence proofs more clearly, it is helpful to work through this special case first.

Definition 6. *(Exchange economy.) An exchange economy consists of m consumers, indexed by the set $I = \{1, ..., m\}$. For each $i \in I$, there is a consumption set $X_i \subset \Re^n$ and an initial endowment $w_i \in \Re^n$.*

Definition 7. *(Allocations and preferences.) An allocation is a point $x = (x_1, ..., x_m) \in \prod_I X_i = X$. Each consumer has a preference relation $P_i \subset X \times X$. For each consumer, define i's Nash improvement preference relation, $\hat{P}_i$ as in Definition 4.*

Preference relations P_i are defined over the allocation space X rather than simply over i's own consumption space X_i. This allows for the possibility of "consumption externalities". This allows for the possibility of benevolence, envy, emulation and other direct

projection of $\hat{P}_i$, while $\hat{P}_i$ is the Cartesian product of P_i^* with "identity" correspondences for each $j \neq i$, propositions 5 and 6 of the Appendix imply that if $\hat{P}_i$ is l. s. c., then so is P_i^* and conversely.

[9] or for that matter in any economy where aggregate demand behaves as if it is the demand of a single utility-maximizing consumer.

effects of others' consumption on consumer preferences. Although consumers may care about the consumption of others, the equilibrium model allows them to choose only their own consumption bundles subject to their own budgets. McKenzie (1955) was the first to extend existence theory to this model. Others who study existence with externalities include Arrow and Hahn (1971) and Shafer and Sonnenschein (1975).

Definition 8. *(Free-disposal competitive equilibrium.) A price vector $\bar{p} > 0$ and an allocation $\bar{x}$ constitute a free-disposal competitive equilibrium if :*

(i) For all $i \in I$, $\bar{p}\bar{x}_i = \bar{p}w_i$.

(ii) For all $i \in I$, $\hat{P}_i(\bar{x}) \cap \{x \in X \mid \bar{p}x_i \leq \bar{p}w_i\} = \emptyset$.

(iii) $\sum_I \bar{x}_i \leq \sum_I w_i$.

A free-disposal competitive equilibrium for an exchange economy is equivalent to competitive equilibrium in a production economy where the only "production" possibilities are disposal activities for every good. In equilibrium supply for some goods can exceed consumers' demand, but any excess supply is removed by the disposal activity. Profit maximization in the "free-disposal industry" requires that equilibrium prices of all goods be nonnegative and prices of goods that are in excess supply must be zero. Since equilibrium prices must be non-negative, conditions (i) and (iii) of equilibrium imply that any good in excess supply must have zero price.[10]

We follow a path first marked out by Debreu (1962), who proved the existence of "quasi-equilibrium" under relatively weak assumptions and then showed that additional assumptions imply that a quasi-equilibrium is a competitive equilibrium.

Definition 9. *(Free-disposal quasi-equilibrium). A price vector $\bar{p} > 0$ and an allocation $\bar{x}$ constitute a free-disposal quasi-equilibrium if conditions (i) and (iii) of Definition 8 are satisfied as well as:*

(ii′) For all $i \in I$, $\hat{P}_i(\bar{x}) \cap \{x \in X_i \mid \bar{p}x_i < \bar{p}w_i\} = \emptyset$.

Theorem 4. *There exists a free-disposal quasi-equilibrium for an exchange economy if:*

4.A) For all $i \in I$, the consumption set X_i is a convex, compact[11] set in $\Re^n$ and $w_i \in X_i$.

4.B) For all $i \in I$, preference relations P_i are l. s. c.

4.C) For all $i \in I$ and for all $x \in X$, $x_i \notin \mathrm{con}\,\hat{P}_i(x)$.

4.D) (Local nonsatiation.) For all $i \in I$, if $x \in X$, then every open neighborhood of x_i in X_i contains a point $x_i' \in \hat{P}_i(x)$.

[10] If the budget constraint in (i) were relaxed to require only that $\bar{p}x_i \leq \bar{p}w_i$, then the weakened definition of equilibrium would not imply that prices of goods in excess supply be zero. On the other hand, if $\bar{p}x_i \leq \bar{p}w_i$ for all i and if all prices are non-negative and goods in excess supply have zero prices, then it must be that in equilibrium, $\bar{p}\bar{x}_i = \bar{p}w_i$ for all i.

[11] The assumption that consumption sets are compact could be replaced by the weaker assumption that these sets are closed and bounded from below. To do so, one would use the truncation technique devised by Debreu (1962).

Let S^n be the unit simplex $\{p \in \Re^n_+ | \sum_{i=1}^n p_i = 1\}$ and let $\hat{X} = S^n \times X$. The following correspondences will with domain $\hat{X}$ will be useful:

Definition 10. *(Price-adjustment correspondence ϕ_p.) Define $\phi_p : \hat{X} \to S^n$ so that*

$$\phi_p(p,x) = \{p' \in S^n | p' \sum_I^n (x_i - w_i) > p \sum_I^n (x_i - w_i)\}.$$

Definition 11. *(Allocation-adjustment correspondence ϕ_i.) For each $i \in I$, define $\phi_i : \hat{X} \to X$*

(i) If $(p,x) \in \hat{X}$ and $px_i \le pw_i$, then $\phi_i(p,x) = \hat{P}_i(x) \cap \{x' | px'_i < pw_i\}$.

(ii) If $(p,x) \in \hat{X}$ and $px_i > pw_i$, then $\phi_i(p,x) = \{x' | px'_i < px_i$ and $x'_j = x_j$ for all $j \neq i\}$.

Definition 12. *(Disequilibrium correspondence ϕ.) Define $\phi : \hat{X} \to \hat{X}$ so that*

$$\phi(p,x) = (\phi_p(p,x) \times x) \bigcup \left(p \times \bigcup_I \phi_i(p,x) \right).$$

Consider a binary relation Q such that $(p',x')Q(p,x)$ if and only if $(p',x') \in Phi(p,x)$. Lemma 3, which is proved in the Appendix, shows that if preferences satisfy the assumptions of Theorem 4, then Q will satisfy the assumptions of Theorem 1 and will have a maximal element. Theorem 4 will then be proved by showing that a maximal element of Q is a free-disposal quasi-equilibrium.

Lemma 3. *Under the assumptions of Theorem 4, the correspondence ϕ is l. s. c. and for all $(p,x) \in \hat{X}$, $(p,x) \notin \text{con}\,\phi(\text{p,x})$. If $\hat{\phi}(p,x) = \emptyset$, then $\hat{\phi}_p(p,x) = \emptyset$ and for all $i \in I$, $\hat{\phi}_i(p,x) = \emptyset$.*

Proof of Theorem 4:

Let Q be the relation on $\hat{X}$ such that $Q(p,x) = \phi(p,x)$. According to Lemma 3, Q satisfies the hypothesis of Theorem 2. Therefore there exists $(\bar{p},\bar{x}) \in \hat{X}$ such that $\phi(\bar{p},\bar{x}) = \emptyset$. This implies that $\phi_p(\bar{p},\bar{x}) = \emptyset$ and for all i, $\phi_i(\bar{p},\bar{x}) = \emptyset$. If $px_i > pw_i$ for some i, then since $w_i \in X_i$, it must be that $\phi_i(p,x) \neq \emptyset$. Since for all $i \in I$, $\phi_i(\bar{p},\bar{x}) = \emptyset$, it follows from the definition of ϕ_i that $\bar{p}\bar{x}_i \le \bar{p}w_i$ and if $x'\hat{P}_i\bar{x}$, then $\bar{p}x'_i \ge pw_i$. Since preferences are assumed to be locally nonsatiated, this can be the case only if $\bar{p}\bar{x}_i = \bar{p}w_i$ for all $i \in I$.

Therefore $\bar{p}\sum_I(\bar{x}_i - w_i) = 0$. Since $\phi_p(\bar{p},\bar{x}) = \emptyset$, it must be that $\bar{p}\sum_I(\bar{x}_i - w_i) \ge p\sum_I(\bar{x}_i - w_i)$ for all $p \in S^n$. Unless $\sum_I(\bar{x}_i - w_i) \le 0$, it would be possible to find $p \in S^n$ such that $p\sum_I(\bar{x}_i - w_i) > 0$. Therefore $\sum_I \bar{x}_i \le \sum_I w_i$ and $(\bar{p},\bar{x})$ must be a quasi-equilibrium. ∎

The existence of a free-disposal competitive equilibrium can be established by adding sufficient assumptions to guarantee that a quasi-equilibrium is a competitive equilibrium. Here we choose assumptions that lead to a very simple proof. As will be shown in the next section, the assumption that initial endowments can be greatly weakened. Assumption 5.B) is weaker than the assumption that "better than" sets are open.

Theorem 5. *There exists a free-disposal competitive equilibrium if preferences satisfy the assumptions 4.1, 4.2 and 4.3 of Theorem 4 as well as*

5.A) For all i, w_i is in the interior of X_i.

5.B) For all i, if $x\hat{P}_i y$ and if $x' \in X$ and $x'_j = x_j$ for $j \neq i$, then there exists a scalar, $\lambda \in (0,1)$, such that $\lambda x + (1-\lambda)x'\hat{P}_i y$.

Proof of Theorem 5:

Suppose that $(\bar{p}, \bar{x})$ is a quasi-equilibrium and $x\hat{P}_i\bar{x}$ for some i. Then it must be that $\bar{p}x_i = \bar{p}w_i$. Since w_i is in the interior of X_i, there exists some $z_i \in X_i$ such that $\bar{p}z_i < \bar{p}w_i$. For $\lambda \in (0,1)$, let $x(\lambda) \in X$ be the allocation such that $x(\lambda)_i = \lambda x_i + (1-\lambda)z$ and for $j \neq i$, $x(\lambda)_j = x_j$. According to Assumption 5.1, there is some λ for which $x(\lambda)\hat{P}_i x$. Since $\bar{p}z_i < \bar{p}w_i$, if $\bar{p}x_i \leq \bar{p}w_i$, it must be that $\bar{p}x(\lambda)_i < pw_i$. But this cannot be since $(\bar{p}, \bar{x})$ is a quasi-equilibrium. Therefore it must be that if $x\hat{P}_i\bar{x}$, then $\bar{p}x_{\geq}\bar{p}w_i$, so $(\bar{p}, \bar{x})$ must be a competitive equilibrium. ∎

Remarks on Quasi-equilibrium and "Compensated Equilibrium".

Debreu (1962) defined "quasi-equilibrium" to be a price vector and an allocation that satisfies all of the conditions of competitive equilibrium except that condition (ii) is weakened to

(ii*) If $\bar{p}\bar{x}_i > \min_{x \in X_i} \bar{p}x$, then $\hat{P}_i(\bar{x}) \cap \{x \in X \mid \bar{p}x_i \leq \bar{p}w_i\} = \emptyset$.

This condition implies that in a quasi-equilibrium, no consumer can find a bundle that is both cheaper and preferred to his competitive bundle. Debreu (1959) showed that quasi-equilibrium exists under weaker assumptions than are needed for competitive equilibrium.

Arrow and Hahn (1971) and Moore (1975) defined a slightly different notion of quasi-equilibrium from Debreu's which they call "compensated equilibrium." They define a compensating equilibrium that has properties (i) and (iii) of competitive equilibrium, while (ii) is replaced by the assumption that for any consumer, anything "at least as good" as his bundle in quasi-equilibrium must cost at least as much.

When preferences are not locally non-satiated, "compensated equilibrium " is not quite the appropriate tool for equilibrium analysis. If there are thick indifference curves, then for some price vectors and for some consumers there may be no commodity bundles such that the consumer spends his entire wealth (Condition (i)) and such that the consumer is also minimizing his expenditure on the set of bundles at least as good as the chosen bundle. This difficulty is remedied by weakening the Arrow-Hahn-Moore definition to replace condition (ii) of the definition of competitive equilibrium by the condition that "no consumer can find a preference bundle is both cheaper and strictly preferred to his equilibrium bundle." This is the definition of "quasi-equilibrium that will be used here. This definition first appeared in Bergstrom (1975) and was adopted by Geistdoerfer-Florenzano (1982), who found it to be advantageous for the same reasons.

5. Equilibrium without Free Disposal, Monotonicity or Local Nonsatiation

If at some allocations, consumers prefer less of some commodities to more, and if there is no free disposal, then we must insist that in competitive equilibrium supply *equals* demand in every market. If this is the case, we must allow the possibility that some commodities have negative prices in competitive equilibrium.

Definition 11. *(Competitive equilibrium.) A price vector* $\bar{p} \neq 0$ *and an allocation* $\bar{x}$ *constitute a competitive equilibrium if :*

(i) For all $i \in I$, $\bar{p}\bar{x}_i = \bar{p}w_i$.

(ii) For all $i \in I$, $\hat{P}_i(\bar{x}) \cap \{x \in X \mid \bar{p}x_i \leq \bar{p}w_i\} = \emptyset$.

(iii) $\sum_I \bar{x}_i = \sum_I w_i$.

Definition 12. *(Quasi-equilibrium.) A price vector* $\bar{p} \neq 0$ *and an allocation* $\bar{x}$ *constitute a quasi-equilibrium if conditions (i) and (iii) of Definition 11 are satisfied as well as:*

(ii′) For all $i \in I$, $\hat{P}_i(\bar{x}) \cap \{x \in X_i \mid \bar{p}x_i < \bar{p}w_i\} = \emptyset$.

Existence of quasi-equilibrium can be proved using similar techniques to those used for proving free-disposal quasi-equilibrium, but without free disposal we will need two extra tricks.

The trick used to find equilibrium when prices may be negative was first used by Bergstrom (1973) and later applied by Bergstrom (1976) and Shafer (1976). Equilibrium prices will be found somewhere on the unit sphere $\{p \in \Re^n | \|p\| = 1\}$. Since the unit sphere is not a convex set, it is mathematically convenient to define correspondences over the unit ball, $B^n = \{p \in \Re^n | \|p\| \leq 1\}$ and to show that the equilibrium that is ultimately found lies on the unit sphere. To do this, we alter the "budget correspondences" for price vectors p such that $\|p\| < 1$ to give each consumer an "income" of $pw_i + \frac{1-\|p\|}{m}$ (where m is the number of consumers in I). As it turns out, our binary relation will have a maximal element only where $\|p\| = 1$, so that in the equilibrium that we establish, incomes are simply pw_i.

In order to do without the assumption of local nonsatiation, we borrow a trick from Gale and Mas Collel. We define an augmented preference relation $\tilde{P}_i$ that includes $\hat{P}_i$ and shares the convexity and continuity properties of $\hat{P}_i$, but is locally non-satiated.

Definition 13. *(Gale-Mas Collel augmented preference relation* $\tilde{P}_i$*.) Define* $\tilde{P}_i \subset X \times X$ *so that* $\tilde{P}_i(x)$ *consists of all points in* $\mathrm{con}\,\{\hat{\mathrm{P}}_\mathrm{i}(\mathrm{x}) \cup \{\mathrm{x}\}\}$ *except for* x *itself.*

Lemma 4.

(i) If $\hat{P}_i(x) \neq \emptyset$, then every neighborhood of x contains a point $x' \in \tilde{P}_i(x)$.

(ii) If $\hat{P}_i$ is l. s. c., so is $\tilde{P}_i$.

(iii) If $x \notin \mathrm{con}\,\hat{\mathrm{P}}_\mathrm{i}(\mathrm{x})$, then $x \notin \mathrm{con}\,\hat{\mathrm{P}}_\mathrm{i}(\mathrm{x})$.

We define a series of correspondences that parallel the constructions for the free-disposal case except that the domain of prices is the unit ball rather than the unit simplex,

Definition 14. *(Price-adjustment correspondence $\tilde{\phi}_p$.) Let $\tilde{X} = B^n \times X$. Define $\tilde{\phi}_p$: $\tilde{X} \to B^n$ so that $\tilde{\phi}_p(p, x) = \{p' | p' \sum_I^n (x_i - w_i) > p \sum_I^n (x_i - w_i)\}$.*

Definition 15. *(Allocation-adjustment correspondence $\tilde{\phi}_i$.) For each $i \in I$, define $\tilde{\phi}_i$: $\tilde{X} \to X$ so that*

(i) If $(p, x) \in \tilde{X}$ and $px_i \leq pw_i + \frac{(1-\|p\|)}{m}$, then $\phi_i(p, x) = \{x' | x' \tilde{P}_i x \text{ and } px'_i < pw_i\}$.

(ii) If $(p, x) \in \tilde{X}$ and $px_i > pw_i$, then $\phi_i(p, x) = \{x' | px'_i < px_i \text{ and } x'_j = x_j \text{ for all } j \neq i\}$.

Definition 16. *(Disequilibrium correspondence $\tilde{\phi}$) Define $\tilde{\phi} : \tilde{X} \to \tilde{X}$ so that*

$$\tilde{\phi}(p, x) = \left(\tilde{\phi}_p(p, x) \times X\right) \bigcup \left(B^n \times \bigcup_I \tilde{\phi}_i(p, x)\right)$$

If we define a relation Q such that xQy if $x \in \phi(y)$, then Lemma 5 ensures that Q takes a maximal element on $\tilde{X}$. A proof of Lemma 5 is found in the Appendix. In the proof of Theorem 6, we show that a maximal element of Q is a quasi-equilibrium.wi

Lemma 5. *The correspondence $\tilde{\phi}$ is l. s. c., for all $x \in \tilde{X}$, $x \notin \text{con}\,\tilde{\phi}(\text{x})$. If $\tilde{\phi}(p, x) = \emptyset$, then $\tilde{\phi}_p(p, x) = \emptyset$ and for all $i \in I$, $\tilde{\phi}_i(p, x) = \emptyset$.*

Theorem 6. *There exists a quasi-equilibrium for an exchange economy if:*

6.A) For all $i \in I$, the consumption set X_i is a convex, compact set, and $w_i \in X_i$.

6.B) For all $i \in I$, preference relations P_i are l. s. c.

6.C) For all $i \in I$ and for all $x \in X$, $x_i \notin \text{con}\,\hat{\text{P}}_\text{i}(\text{x})$.

6.D) (No feasible bliss points.) If $\sum_I (x_i - w_i) = 0$, then for all $i \in I$, $\hat{P}_i(x) \neq \emptyset$.

Proof of Theorem 6:

According to Lemma 5, the correspondence $\tilde{\phi}$ satisfies the hypothesis of Theorem 2. Therefore there exists $(\bar{p}, \bar{x}) \in \tilde{X}$ such that $\tilde{\phi}(\bar{p}, \bar{x}) = \emptyset$.

Since $\phi(\bar{p}, \bar{x}) = \emptyset$, it must be that $\bar{p} \sum_I (\bar{x}_i - w_i) \geq p \sum_I (\bar{x}_i - w_i)$ for all $p \in B^n$. This implies that either $\sum_I (\bar{x}_i - w_i) = 0$, or $\bar{p} = \frac{\sum_I (\bar{x}_i - w_i)}{\| \sum_I (\bar{x}_i - w_i) \|}$, in which case $\|\bar{p}\| = 1$. Therefore if $\|\bar{p}\| < 1$, $\sum_I (\bar{x}_i - w_i) = 0$. But if $\sum_I (\bar{x}_i - w_i) = 0$, then by Assumption 6.D, for all i, $\tilde{P}_i(\bar{x}) \neq \emptyset$. Suppose That $\bar{p}\bar{x}_i < \bar{p}w_i + \frac{1-\|p\|}{m}$ for some i. Then from Lemma 4(i), it follows that there must be $x' \in \tilde{P}_i(\bar{x})$ such that $\bar{p}x'_i < \bar{p}w_i$. but this cannot be since $\tilde{\phi}_i(\bar{p}, \bar{x}) = \emptyset$. Therefore $\bar{p}\bar{x}_i \geq \bar{p}w_i + \frac{1-\|p\|}{m}$ for all $i \in I$ and hence $\bar{p} \sum \bar{x}_i - \bar{p}w_i \geq 1 - \|p\| > 0$. But this is impossible since $\sum \bar{x}_i - w_i = 0$. We must conclude that $\|\bar{p}\| = 1$.

Given that $\|p\| = 1$ and that $\tilde{\phi}_i(\bar{p}, \bar{x}) = \emptyset$, it must be that $\bar{p}\bar{x}_i \leq \bar{p}w_i$ for all $i \in I$ and $\bar{p} \sum_I (\bar{x}_i - w_i) \leq 0$. Since $\phi_0(\bar{p}, \bar{x}) = \emptyset$, and $\bar{p} \sum_I (\bar{x}_i - w_i) \leq 0$, it must be that $\sum_I (\bar{x}_i - w_i) = 0$. Since $\phi_i(\bar{p}, \bar{x}) = \emptyset$, it must also be that $\tilde{P}_i(\bar{x}) \cap \{x | \bar{p}x_i < \bar{p}w_i\} = \emptyset$ and hence $\hat{P}_i(\bar{p}, \bar{x}) = \emptyset$ for all $i \in I$. But this implies that $(\bar{p}, \bar{x})$ is a quasi-equilibrium. ∎

A quasi-equilibrium can be shown to be competitive equilibrium under much weaker assumptions than Debreu's (1959) assumption that every consumer has a positive intial endowment of every good. Debreu's assumption is used to show that at any possible equilibrium prices, each consumer can always afford some bundle that costs strictly less than his budget constraint. A quasi-equilibrium will have this property if each consumer could survive after having surrendered some vector of commodities that could be used to make allwould make all other consumers better off. This idea is embodied in the "irreducibility" assumptions of McKenzie (1959), (1961) and Debreu (1962) and the "resource-relatedness" assumptions of Arrow and Hahn (1971). The conflict of interest assumption used here is taken from Bergstrom (1976). Geistdoerfer-Florenzano (1982) discusses these alternative assumptions in detail and shows that our conflict of interest assumptions (1976) assumptions are weaker than either "McKenzie-Debreu irreducibility" or "Arrow-Hahn irreducibility".[12]

Definition 17. *(Conflict of interest.) Let $P_i^*(x)$ be the projection of $\hat{P}_i(x)$ as in Definition 4. There is conflict of interest if for any allocation $x \in X$ such that $\sum_I (x_i - w_i) = 0$ and if for consumer $j \in I$, $w_j \notin P_j^*(x)$ then there exists $x' \in X$ and numbers $\theta_i > 0$ such that for all $i \neq j$, $x'_i \in P_i^*(x)$ and such that $\sum_{i \in I} \theta_i (x'_i - w_i) = 0$.*

If the conflict of interest condition holds with the θ_i's all equal, then for any j and any feasible allocation x that leaves consumer j no worse off than he would be without trading, there exists another feasible allocation that is better for all consumers other than j. Allowing the possibility that θ_i's may differ means that there is also conflict of interest if for some arbitrarily chosen numbers of persons of each type in the economy, it would be possible for the persons of all types other than j to be made better off (typically at the expense of the type j's.

Proof of Theorem 7:

A quasi-equilibrium $(\bar{p}, \bar{x})$ is a competitive equilibrium if :

7.A) For all i, if $x\hat{P}_i y$ and if $x' \in X$ and $x'_j = x_j$ for $j \neq i$, then there exists a scalar $\lambda \in (0,1)$, such that $\lambda x + (1-\lambda) x' \hat{P}_i y$.

7.B) There is conflict of interest.

7.C $\sum_I w_i \in \text{Interior} \sum_I X_i$.

Proof:

Let $K = \{i \in I | \bar{p}x_i < pw_i \text{ for some } i \in I\}$. From Assumption 7.C and the fact that $\sum_I (\bar{x}_i - w_i) = 0$, it follows that K is not empty. Using Assumption 7.A, it is easy to show that for $i \in K$, $\hat{P}_i(\bar{x}) \cap \{x \in X \mid \bar{p}x_i \leq \bar{p}w_i\} = \emptyset$. Suppose that for some $i \in I$, $i \notin K$. By assumption 7.B, for all $j \neq i$, there exists $x'_j \in P_j^*(x)$ and $\sum_I (x'_i - w_i) = 0$. Since $(\bar{p}, \bar{x})$ is a quasi-equilibrium, it must be that for all $j \neq i$, $\bar{p}x'_j \geq \bar{p}w_j$ and since K is not

[12] Moore's (1982) assumption that "Each consumer is productive" is more similar to our assumption than the McKenzie-Debreu of Arrow-Hahn, but slightly stronger.

empty, it must be that for some $k \neq i$, $\bar{p}x'_k > \bar{p}w_k$. Therefore $\bar{p}\sum_{j \neq i}(x'_j - w_j) > 0$. But $\bar{p}\sum_I(x'_j - w_j) = 0$, since $\sum_I(x'_j - w_j) = 0$. Therefore $\bar{p}x'_i < \bar{p}w_i$. But this contradicts the assumption that $i \notin K$. It follows that $K = I$ and hence $\hat{P}_i(\bar{x}) \cap \{x \in X \mid \bar{p}x_i \leq \bar{p}w_i\} = \emptyset$ for all $i \in I$. ∎

Adding Production to the Model

One might think that adding production to this model would be a chore, but Trout Rader has shown how to make this task easy. In his Ph. D. thesis, published in *Yale Economic Essays* in 1963, Rader showed that a pure exchange economy can be interpreted as an economy with household production. This is done by defining "induced preferences" on net trades in the natural way, given the consumer's initial endowment and his household production possibility set. Rader further shows that the standard Arrow-Debreu model of a "private ownership economy" with convex production possibility sets can be incorporated into this framework. Where consumer i owns fraction θ_{ij} of firm j which has production possibility set Y_j, the Arrow-Debreu model is equivalent to an exchange model in which each consumer i has a household production possibility set, $\sum_j \theta_{ij}Y_j$. In an unpublished paper, Rader (1975) shows that these ideas carry over nicely to the case of preferences that need not be transitive or complete.

Nonconvex consumption sets and consumers who can't survive without trade.

We have have assumed that consumption sets are closed, convex and that each consumer's initial endowment lies in his consumption set. Moore (1975) argued that for a modern economy it is unrealistic to assume that consumers can survive without trade. Moore proves the existence of quasi-equilibrium without assuming $w_i \in X_i$ for a model with transitive, locally nonsatiated preferences. Moore uses an assumption that each consumer is "productive in the economy", which is very similar to our "conflict of interest" assumption. McKenzie (1981) extended Moore's results to the the case of preferences which need not be transitive or locally nonsatiated.[13]

While it is probably true that in a modern economies, many consumers would not live long without trade, it is not obvious that the "consumption sets" of equilibrium theory should be identified with sets of commodity bundles that allow a person to "survive". In fact in an economy with dated commodities, there is some question of what is meant by survival. Few consumers would die immediately if they were unable to make any trades. Regardless of what trades they make, real-world consumers do not live forever.

In an exchange economy where tradables consist only of an existing stock of n "goods", it seems reasonable to let each consumer's consumption set be the entire non-negative orthant in $\Re^n$. Bundles of goods can be delivered or collected from each consumer's "warehouse", subject only to the constraint that warehouses always contain nonnegative

[13] McKenzie regard's Moore's weakening of the survial assumption as "perhaps the most dramatic innovation since 1959" in the theory of existence of competitive equilibrium.

stocks of each commodity.[14] This is true even if, after all deliveries and collections are made, some consumers are left with such meager bundles that they can not "survive". If each consumer has a nonnegative initial endowment, then $w_i \in X_i$ for all i, since this is just the outcome if no deliveries or collections occur. With this interpretation, it might not be reasonable to assume local nonsatiation or convex preferences over consumption bundles that lead to early starvation.[15] But the assumptions of the model considered here, which assumes neither local nonsatiation nor convexity of preferences, presents no difficulties for this interpretation.

In an economy where people exchange labor services for goods, and where commodities are distinguished by time of delivery as well as by type, it may not be appropriate to identify the consumption set with the nonnegative orthant. Feasible combinations of food input, years of survival and labor supply must be related in a fairly complicated way. Matters are clarified if we specify the set of "feasible trades" quite independently of the notion of survival. The set of feasible trades for a consumer with consumption set X_i and endowment w_i is the set $X_i - w_i$. Even if a consumer can not "survive" without trade, zero trade is a possible outcome and hence we can reasonably assume that $w_i \in X_i$. If zero trade leads to early demise, the problem that arises for equilibrium theorists is that it is not reasonable to assume that the set of feasible trades is convex.

For example, consider an economy with two commodities, bread and labor, which can be made available in two different time periods. A consumer has no bread in his initial endowment and must have at least one unit of bread in the first period in order to survive to the second period. If he doesn't survive, then he can offer no labor in the second period. One feasible outcome has the consumer making no trade in either period and starving to death before the second period. Another possibility is that he offers one unit of labor in each time period and receives one unit of bread in each period. For the set if feasible trades to be convex, it must be feasible to supply half a unit of labor in each time period and consume half a unit of bread in each period. But we have assumed that if he gets less than one unit of bread in the first period, then he will be unavailable to supply any labor in the second period. Therefore the set of feasible trades is not convex.

Survival sets and feasible consumptions sets are distinguished as follows. For each consumer i, let there be a "survival set" S_i and a set of "feasible consumptions" $X_i \supset S_i$. The survival set can be interpreted as the set of commodity bundles that allow the consumer to reach some threshold of general health. As a formal matter, the set S_i is assumed to be convex and closed, while the set X_i is compact but need not be convex. We will also assume that consumer i never prefers consumption bundles not in $X_i \sim S_i$ to bundles in S_i.

We can now modify Theorem 6 on the existence of quasi-equilibrium to allow the possibility that the set of feasible trades is not convex.

Theorem 6′. *There exists a quasi-equilibrium for an exchange economy if Assumptions*

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[14] The existence theorem in this paper permits the possibility that warehouses are of bounded capacity.

[15] With convex preferences, if $x \, P \, y$ then $\lambda x + (1 - \lambda) y \, P \, y$. If consumptions below a certain threshold all lead to starvation and are equally bad, then this assumption would not be reasonable.

6.B, 6.C, and 6.D are satisfied and also

6.A') Every consumer i has a compact consumption set X_i and a closed, convex survival set $S_i \subset X_i$ such that if $y \in S_i$ and $x \in X_i \sim S_i$, then not xP_iy.

With this weakened assumption, the proof of Theorem 6 goes through almost exactly as before. The correspondence $\phi(p, x)$ still satisfies conditions 2.A) and 2.B) of Theorem 2. Since X is not convex, we can not apply Theorem 2 directly, but the corollary that appears directly after Theorem 2 implies that $\phi(\bar{p}, \bar{x}) = \emptyset$ for some $(\bar{p}, \bar{x})$. The rest of the proof works as before.

Under the assumptions of Theorem 7, the proposition that a quasi-equilibrium must be a competitive equilibrium remains true with the weakened assumptions of Theorem $6'$.

Appendix—Properties of Lower Semi-Continuous Correspondences

Here we report several useful properties of l. s. c. correspondences. Those propositions that are stated without proofs all have simple proofs which are reported elsewhere. See Berge (1963) and Michael (1956).

Proposition 1 simply restates the definition of lower semi-continuity in a useful way.

Proposition 1. *The correspondence $\phi : X \to 2^Y$ in l. s. c. iff for all $x \in X$, $y \in \phi(x)$ implies that for every open neighborhood V of y in Y there exists an open neighborhood U of x in X such that for all $x' \in U$, $\phi(x') \cap V \neq \emptyset$.*

An immediate consequence of Proposition 1 is the fact that a sufficient (but not necessary) condition for a correspondence to be l. s. c. is that the "inverse images of points" are open sets.

Proposition 2. *Let S be a topological space, $P \subset S \times S$ and $X \subset S$. If for all $x \in S$, $P^{-1}(x)$ is open in X, then P is l. s. c. on X.*

Proof. Let V be an open subset of S. Then $\{x \in X \mid P(x) \cap V \neq \emptyset\} = \bigcup_{y \in V} (P^{-1}(y) \cap X)$ which is a union of sets which are open in X and hence is open in X. Therefore P is l. s. c. on X.∎

Unions and finite intersections of l. s. c. correspondence must also be l. s. c.

Proposition 3. *For any index set I, let $\phi_i : X \to Y$ be l. s. c. for all $i \in I$. Then $\Psi : X \to Y$ is l. s. c. where $\Psi(x) = \bigcup_{i \in I} \phi_i(x)$.*

Proposition 4. *For any finite index set I, let $\phi_i : X \to Y$ be l. s. c. for all $i \in I$. Then $\Psi : X \to Y$ is l. s. c. where $\Psi(x) = \bigcap_{i \in I} \phi_i(x)$.*

Projections and Cartesian products of l. s. c. correspondences must also be l. s. c.

Proposition 5. *Let I be a finite index set and $X = \prod_{i \in I} X_i$. Let $\phi : X \to X$ be l. s. c. and let $\phi_i : X \to 2^{X_i}$ to be the projection of ϕ on X_i. Then ϕ_i is l. s. c.*

Proposition 6. *Let I be a finite index set and $X = \prod_{i \in I} X_i$ where such X_i is a topological space. Let $\phi_i : X \to 2^{X_i}$ be l. s. c. for all $i \in I$. Let $\phi : X \to X$ be defined so that $\phi(x) = \prod_{i \in I} \phi_i(x)$. Then ϕ is l. s. c.*

Proposition 7. *Let $X = \prod_{i \in I} X_i$. Let $\phi_i : X \to X_i$ be l. s. c. Let $\hat{\phi}_i : X \to X$ be defined so that $\hat{\phi}_i(p, x) = \{x' \in X | x'_i \in \phi_i(x)$ and $x'_j = x_j$ for all $j \neq i\}$.Then $\hat{\phi}_i$ is l. s. c.*

Proof The correspondence $\hat{\phi}_i$ is just the Cartesian product of ϕ_i with the "identity correspondences" that maps $x_j \in X_j$ into the set $\{x_j\}$ for each $j \neq i$. These correspondences are all l. s. c. According to Proposition 6, the Cartesian product of l. s. c.. correspondences is l. s. c.. ∎

Proposition 8. *Let* $X = \prod_{i \in I} X_i$. *Let* $\phi : X \to X$ *be l. s. c. For* $i \in I$, *define* $\hat{\phi}_i : X \to X$ *so that* $\hat{\phi}_i(x) = \{x' | x' \in \phi(x)$ *and* $x'_j = x_j$ *for all* $j \neq i\}$. *Then* $\hat{\phi}_i$ *is l. s. c.*

Proof: For $j \in I$, let $T_j(x) = \{x' \in X | x'_j = x_j\}$. The correspondences T_j are easily seen to be l.s.c. Now $\hat{\phi}_i(x) = \phi(x)_{j \neq i} T_j$. Since finite intersections of l. s. c.. correspondences are l. s. c.., $\hat{\phi}_i$ is l. s. c.. ∎

If a correspondence is lower semi-continuous, then the correspondence whose image sets are the convex hulls of the image sets in the original correspondence will also be l. s. c.

Proposition 9. *If* Y *is a linear topological space and* $\phi : X \to 2^Y$ *is l. s. c., then the correspondence,* Ψ, $X \to 2^Y$ *where* $\Psi(x) = \text{con}\, \phi(\text{x})$ *is l. s. c.*

The next result is useful for establishing existence of equilibrium when preferences are not monotonic and there is no free disposal.

Proposition 10. *Let* $\phi_i : \prod X_i \to X_i$ *be l. s. c. and define* $\phi_i^* : \prod X_i \to X_i$ *so that* $\phi_i^*(x) = \{z \in X_i \mid z = \lambda y_i + (1 - \lambda) x_i$ *where* $y_i \in \phi_i(x)$ *and* $0 < \lambda \leq 1\}$. *Then* ϕ_i^* *is l. s. c.*

Proof. Consider the correspondence $\Psi_i(x) = \phi_i(x) \cup I_i(x)$, where $I_i(x)$ is the ith projection of the identity correspondence. Then Ψ_i must be l. s. c., since it is the union of two l. s. c. correspondences. Where ϕ_i^* is the correspondence defined in Proposition 7, $\phi_i^*(x) = \text{con}\,\text{Psi(x)}$ for all x and hence ϕ_i^* must be l. s. c., according to Proposition 6.

An alternative characterization of lower semi-continuous correspondences is in terms of sequences.

Proposition 11. *Where* X *and* Y *satisfy the first axiom of countability*[16] *, the following property is necessary and sufficient that* $\phi : X \to 2^Y$ *be l. s. c. If* $y \in \phi(x)$ *and* $x(n) \to x$, *there exists a subsequence* $x(n_k) \to x$ *such that for some sequence,* $y(k)$ *in* Y, $y(k) \to y$ *and* $y(k) \in \phi(x(n_k))$ *for every integer* k.[17]

Proof.

Suppose that ϕ is l. s. c. Let $y \in \phi(x)$ and let $\{V_k \mid k = 1, 2, ...\}$ be a countable base at y. For each k, let $\hat{V}_k = \bigcap_{i \leq K} V_i$. Since ϕ is l. s. c., for each k there exists a neighborhood

16 A topological space satisfies the first axiom of countability if the neighborhood system if every point has a countable base. This is true of all metric spaces. If X and Y lack this property, then lower semi-continuity may be characterized by the convergence of nets rather than sequences. This situation is in close analogy to the characterization of continuous functions by sequences or nets. See Kelley (1963).

17 The reader may be familiar with the following characterization of lower semi-continuity. Where $y \in \phi(x)$ and $x(n) \to x$, there exists a sequence $y(n) \to y$ such that $y(n) \in \phi(x(n))$ for all n. If the image sets are always non-empty and X and Y satisfy the first axiom of countability, then this property is necessary and sufficient that ϕ be l. s. c. as defined here. If $\phi(x)$ can be empty this property is not necessary. (In particular it may be that $\phi(x(n)) = \emptyset$ for some n.)

U_k of x such that for all $x' \in U_k$, $\hat{V}_k \cap \phi(x') \neq \emptyset$. Let $\hat{U}_k = \bigcap_{i \leq k} U_i$. Since $x(n) \to x$, there exists a subsequence $(x(n_k))$ such that for every k, $x(n_k) \in \hat{U}_k$. Then there exists $y(k) \in \hat{V}_k \cap \phi(x(n_k))$. Since $\{\hat{V}_k \mid k = 1, 2, ...\}$ is a base at y, $y(k) \to y$. Thus $(x(n_k))$ is a subsequence of $x(n)$ such that there exists a sequence $(y(k))$ in Y where $y(k) \to y$ and $y(k) \in \phi(x(n_k))$ for all k. This proves necessity.

Suppose that ϕ is not l. s. c. Then there exist $x \in X$, $y \in Y$ and a neighborhood V of Y such that for every neighborhood U of x, there exists $x' \in U$ such that $\phi(x') \cap V = \emptyset$. Let $\{U_n \mid n = 1, 2, ...\}$ be a countable base at x and for each k, let $\hat{U}_n = \bigcap_{i \leq n} U_i$. Then there exists a sequence $(x(n))$ such that for all n, $x(n) \in \hat{U}_n$ and $\phi(x(n)) \cap V = \emptyset$. Clearly $x(n) \to x$. But there is no subsequence $(x(n_k)$ of $(x(n))$ for which there exists a sequence $(y(k))$ where $y(k) \to y$ and $y(k) \in \phi(x(n_k))$ for every k. This proves sufficiency. ∎

Proof of Lemma 3.

For all $p \in S^n$ and $z \in \Re^n$, the set $\{p' \in S^n | p'z > pz\}$ is open in S^n. It follows that for all $(p, x) \in \hat{X}$, $\phi_p^{-1}(p, x)$ is open in S^n. From Proposition 2, it follows that ϕ_p is l. s. c.

Let $B_i(p, x) = \{x' \in X | px'_i < pw_i\}$. Let $C_i(p, x) = \{x' \in X | px'_i < px_i$ and $x'_j = x_j$ for all $j \neq i\}$. From Proposition 2 it follows that $B_i(p, x)$ is l. s. c.. From Propositions 2 and 8, it follows that $C_i(p, x)$ is l. s. c..

Let $\phi_i^a(p, x) = \hat{P}_i(x) \cap B_i(p)$ if $px_i \leq pw_i$ and $\phi_i^a(p, x) = \emptyset$ if $px_i > pw_i$. Let $\phi_i^b(p, x) = C_i(p, x)$ if $px_i > pw_i$ and $\phi_i^b(p, x) = \emptyset$ if $px_i \leq pw_i$. From Proposition 2 it follows that ϕ_i^a and ϕ_i^b are l. s. c.. Since for all $(p, x) \in \hat{X}$, $\phi_i(p, x) = \phi_i^a \cup \phi_i^b(p, x)$, it follows from Proposition 3 that ϕ_i is l. s. c.. From Proposition 3 it then follows that $\phi = \cup_I \phi_i$ is l. s. c..

From the definition of ϕ, it is immediate that if $\hat{\phi}(p, x) = \emptyset$, then $\hat{\phi}_p(p, x) = \emptyset$ and for all $i \in I$, $\hat{\phi}_i(p, x) = \emptyset$. ∎

References

Aliprantis, C., and D. Brown (1983) “Equilibria in Markets with a Riesz space of commodities,” *Journal of Mathematical Economics*, **11**, 189-207.

Arrow, K., and F. Hahn (1971) *General Competitive Analysis.* San Francisco: Holden-Day.

Berge, C. (1963) *Topological Spaces.* New York: Macmillan.

Bergstrom, T. (1973) “Competitive Equilibrium Without Transitivity, Local Nonsatiation or Monotonicity,” Washington University, St. Louis, MO..

Bergstrom, T. (1975) “The Existence of Maximal Elements and Equilibria in the Absence of Transitivity,” Washington University, St. Louis, Mo..

Bergstrom, T. (1975) “Maximal Elements of Acyclic Relations on Compact Sets,” *Journal of Economic Theory*, **10**, 403-404.

Bergstrom, T. (1976) “How to Discard Free Disposability—At No Cost,” *Journal of Mathematical Economics*, **3**, 131-134.

Bergstrom, T., R. Parks, and T. Rader, (1976) “Preferences Which Have Open Graph,” *Journal of Mathematical Economics*, **3**, 265-268.

Birchenhall, C. (1977) “Conditions for the Existence of Maximal Elements in Compact Sets,” *Journal of Economic Theory*, **16**, 111-115.

Borglin, A. and H. Keiting (1976) “Existence of Equilibrium Actions and of Equilibrium,” *Journal of Mathematical Economics*, **3**, 313-316.

Blume, L. (1986) “,” in *Contributions to Mathematical Economics in Honor of Gerard*

Debreu, ed. Hildenbrand, W. and A. Mas Collel. New York: Elsevier.

Debreu, G. (1959) *The Theory of Value.* New York: Wiley.

Debreu, G. (1962) "New Concepts and Techniques in General Equilibrium Analysis," *International Economic Review*, **3**, 257-273.

Fan, K. (1961) "A Generalization of Tychonoff's Fixed Point Theorem," *Math Annalen*, **142**, 305-310.

Gale, D. and A. Mas Collel (1975) "A Short Proof of Existence of Equilibrium Without Ordered Preferences," *Journal of Mathematical Economics*, **2**, 9-15.

Gale, D. and A. Mas Collel (1979) "Corrections to an Equilibrium Existence Theorem for a General Equilibrium Model Without Ordered Preferences," *Journal of Mathematical Economics*, **6**, 297-299.

Geistdoerfer-Florenzano, M. (1982) "The Gale-Nikaido-Debreu Lemma and the Existence of Transitive Equilibrium with or without the Free-Disposal Assumption," *Journal of Mathematical Economics*, **9**, 113-134.

Kelley, J. (1963) *General Topology.* Princeton, N.J.: Van Nostrand.

McKenzie, L. (1955) "Competitive Equilibrium with Dependent Consumer Preferences," in *Second Symposium on Linear Programming*, ed. . Washington: National Bureau of Standards and Department of the Air Force.

McKenzie, L. (1959) "On the Existence of General Equilibrium for a Competitive Market," *Econometrica*, **27**, 54-71.

McKenzie, L. (1961) "On the Existence of General Equilibrium–Some Corrections," *Econometrica*, **29**, 247-248.

McKenzie, L. (1981) "The Classical Theorem on Existence of Competitive Equilibrium," *Econometrica*, **49**, 819-842.

Mehta, G. (1987) "Maximal Elements of Preference Maps," Working Paper, University of Queensland, Department of Economics, Brisbane, Australia.

Mehta, G. (1987) "Weakly Lower Demicontinuous Preference Maps," *Economics Letters*, **23**, 15-18.

Moore, J. (1975) "The Existence of "Compensated Equilibrium" and the Structure of the Pareto Efficiency Frontier," *International Economic Review*, **16**, 267-300.

Michael, E. (1956) "Continuous Selections, I," *Annals of Mathematics*, **63**, 361-382.

Nash, J. (1950) "Equilibrium States in N-person Games," *Proceedings of the National Academy of Sciences of the U.S.A.*, **36**, 48-49.

Rader, T. (1964Œ) "Edgeworth Exchange and General Economic Equilibrium," *Yale Economic Essays*, **4**, 133-18-.

Rader, T. (1972) *Theory of General Economic Equilibrium.* New York: Academic Press.

Rader, T. (1975) "Induced Preferences on Trades When Preferences May be Intransitive and Incomplete," Washington University, St. Louis, MO..

Schofield, N. (1983) "Social Equilibrium and Cycles on Compact Sets," *Journal of Economic Theory*, **33**, 59-71.

Schofield, N. (1984) "Existence of Equilibrium on a Manifold," *Mathematics of Operations Research*, **9**, 545-557.

Shafer, W. (1974) "The Nontransitive Consumer," *Econometrica*, **42**, 913-919.

Shafer, W. and H. Sonnenschein (1975) "Equilibrium in Abstract Economies without Ordered Preferences," *Journal of Mathematical Economics*, **2**, 345-348.

Shafer, W. and H. Sonnenschein (1976) "Equilibrium with Externalities, Commodity Taxation, and Lump Sum Transfers," *International Economic Review*, **17**, 601-611.

Shafer, W. (1976) "Equilibrium in Economies without Ordered Preferences or Free Disposal," *Journal of Mathematical Economics*, **3**, 135-137.

Sonnenschein, H. (1971) "Demand Theory Without Transitive Preference with Applications to the Theory of Competitive Equilibrium," in *Preferences, Utility and Demand*, ed. Chipman, J., L. Hurwicz, M. Richter, and H. Sonnenschein, eds.. New York: Harcourt, Brace, Jovanovich.

Walker, M. (1977) "On the Existence of Maximal Elements," *Journal of Economic Theory*, **16**, 470-474.

Yannelis, N. and N. Prabhakar (1983) "Existence of Maximal Elements and Equilibria in Linear Topological Spaces," *Journal of Mathematical Economics*, **12**, 233-245.