Concordance among Holdouts∗
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Abstract

A holdout problem arises when a good owned by disparate sellers is desired by a buyer only in its entirety, as in land assembly. Market design is crucial in the presence of holdout, as fully respecting property rights of many sellers eliminates otherwise-efficient trade. We propose a Concordance principle, inspired by the Cournot (1838) theory of concours de producteurs: profits should be divided by exogenously determined shares, and externalities should be internalized through Pigouvian taxation. Concordance alleviates the holdout problem by ensuring protection of collective and approximate individual property rights while, asymptotically, achieving full efficiency. Combining the Concordance principle with standard auction procedures encourages truth-telling, with familiar tradeoffs between implementability and budget-balance. Extensions of our approach yield mechanisms for collaboration and public goods problems.

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Very few commodities are consumed in just the form in which they are left in the hands of the first producer... Several raw materials are generally brought together in the manufacture of each of these products... The more there are of articles thus related, the higher the price determined by the division of monopolies will be, than that which would result from the fusion or association of monopolists.


Cournot (1838) established perhaps the most celebrated theorem in the theory of industrial organization: sufficient competition (*concurrence*) among the producers of substitutable goods leads to efficient price-taking behavior.¹ Less widely known, but nearly as important, is Cournot’s treatment of collaboration (*concours*) among the producers of perfectly complementary pieces of a good described in the quote above.²³ As Sonnenschein (1968) argued, *Cournot collaboration* is the dual of Cournot competition: just as an increase in the number of competitors drives down mark-ups and increases production, increased subdivision of a good into monopolized components drives down production and raises prices. Thus, if each piece of a good that a buyer desires in its entirety is owned by a different self-interested seller, all possibility of beneficial trade (in the absence of expropriation or subsidies) disappears as the number of sellers grows (Bergstrom, 1978; Mailath and Postelwaite, 1990).⁴

This basic holdout problem—no seller is likely to scuttle the sale, each seller can demand all gains from trade—arises in many practical settings: land assembly, corporate acquisitions, patent pool formation, spectrum management, and class action settlement.⁵ Unlike in the classical auction setting, where increasing competition is often more important than design

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¹We detail Cournot’s theory and its relationship to the holdout problem in an online appendix.
²“Collaboration” is the word most commonly given as the best translation of *concours* in the classical sense Cournot used it, and thus we have adopted it despite the contrasting original English translation by Nathaniel T. Bacon (Cournot, 1897).
³This problem is often known as “double marginalization” (Spengler, 1950) or “anti-commons” (Michelman, 1967).
⁴Eliminating strategic incentives for sellers to misreport their values by having a buyer make a take-it-or-leave-it offer is insufficient to ensure efficiency in this setting, as the buyer must still be willing to offer each seller her highest possible value.
⁵These and other applications are discussed in Section IV and in an extensive appendix accompanying this paper. Relatively conservative back-of-the-envelope calculations in Subsection IV.E suggest that deadweight loss from inefficiencies in these applications is on the order of many trillions of dollars, net present value.
(Bulow and Klemperer, 1996; Klemperer, 2002), there is no easy way around the necessity of some social engineering to solve holdout. However, most practical mechanisms used or proposed to solve the holdout problem involve a form of weighted voting. These mechanisms are either unprotective of property rights or highly inefficient. Yet despite an enormous literature on auction design beginning with Myerson (1981), we are not aware of a single formal market design paper proposing a novel solution to the holdout problem. This article begins to fill that gap.

Section II develops a general model of the holdout problem. We then, in Section III, present our primary contribution: inspired by Cournot’s argument that both public and private interests are served by merging collaborating producers into a single firm, we propose a principle underlying our approach to solving holdout. In merged firms, divisions typically share revenue according to predetermined formulae and internalize externalities they cause. The natural extension of Cournot’s insight to the holdout problem is therefore that each seller should have an option to choose not to influence the sale decision and receive at worst a pre-determined share of the buyer’s offer (whenever a sale takes place). Meanwhile, sellers who choose to influence the sale decision should internalize their externalities on other sellers through a Pigouvian tax. In honor of Cournot’s theory, we refer to this principle by the English word directly derived from concours: Concordance.6

While no mechanism can achieve full efficiency while perfectly protecting individual property rights, we show in Section III that any mechanism based on the Concordance principle strikes an attractive balance between these goals. Concordance mechanisms are

- **bilaterally efficient** (always as efficient as bilateral bargaining) and

- **asymptotically efficient** (fully efficient as the number of sellers grows large).

Additionally, Concordance mechanisms protect

- **community property rights** (the sellers as a whole are not harmed) and

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6Of course the root of “concordance,” “concord,” also derives from concours, but we think “concordance” better captures the active role Concordance mechanisms play in achieving efficient outcomes.
• **approximate individual property rights** (each individual receives as compensation at least a natural approximation to their value based on all but their own information).

Unlike eminent domain, implementing Concordance does not require that the government know sellers’ subjective valuations; rather, the government needs only an approximation of each seller’s share of the total community value.7

Specific Concordance mechanisms (Subsections V.A-B) use standard concepts from auction theory to encourage truthfulness. The simplest enforcement strategy is the *Straightforward Concordance* (SC) mechanism of Subsection V.A, based upon the tax scheme suggested by Vickrey (1961), Clarke (1971) and Groves (1973) (VCG): each seller pays a Pigouvian tax for any negative externalities her valuation report causes others. Extending ideas of Green and Laffont (1977) shows that this is the only Concordance enforcement approach that makes truthful reporting dominant for sellers. SC has the typical disadvantage of VCG mechanisms: taxes are lost to a central authority.8

SC can be improved upon if externalities are assessed on some basis other than other sellers’ reports (Arrow, 1979; d’Aspremont and Gérard-Varet, 1979). We refer to the mechanism in which each seller pays her expected externality, rather than her actual externality, as the *Bayes-Nash Concordance* (BNC) mechanism (Subsubsection V.B.1). While BNC is not entirely straightforward, BNC implements truthfulness in Bayes-Nash equilibrium. Furthermore, unlike in SC, tax refunds in BNC can fully balance the budget.

Unfortunately, without a specific, known joint distribution of valuations, there is no reliable way to translate valuations into BNC taxes. Thus, perhaps the best practical approximation to the BNC taxation scheme is to have each seller pay her reported surplus, as an approximation of her expected externality. This approach gives rise to the *All-pay Concordance* (APC) mechanism (Subsubsection V.B.2), which is roughly analogous to the

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7 The quality of this share approximation determines the degree to which individual property rights are upheld. In particular, if the shares are perfectly assessed, then individual property rights are fully protected.

8 We propose a tax refund system that mitigates this problem; adapting the suggestion of Cavallo (2006) to our context, each seller receives her share of the minimum taxes she could cause other sellers to pay. Nonetheless, potential for manipulation and budget surpluses persist.
standard all-pay auction.\(^9\)

The holdout problem is closely related to the large literature on mechanism design for the provision of public goods; In fact, (Subsection VI.A), it is nearly equivalent to a binary public goods game.\(^10\) Despite this, research on the design of practical institutions for handling holdout problems has been severely limited.\(^11\) As far as we know, every mechanism thus far proposed suffers from at least one of three serious flaws. Many mechanisms Nash-implement Lindahl allocations (Groves and Ledyard, 1977; Hurwicz, 1977; Walker, 1981; Bagnoli and Lipman, 1989; Tian, 1989). However, all of these have impractically large multiplicities of equilibria or assume complete information and thus effectively rely on an implausible common knowledge of values to achieve efficiency (Bailey, 1994). Property-rights-preserving mechanisms that merely eliminate strategic misrepresentation of valuations (Bagnoli and Lipman, 1988; Shapiro and Pincus, 2007) do not solve the more fundamental holdout problem.\(^12\) Classic market value-based takings and more efficient proposals for simple Vickrey-Clarke-Groves procedures (Plassmann and Tideman, 2009) abrogate all semblance of property rights.

Thus we view the problem of designing a practical mechanism for the holdout problem that strikes a reasonable compromise between the competing goals of efficiency and fairness (property rights) to be almost entirely open.\(^13\) In Subsection V.D we discuss the general

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\(^9\)An equivalent implementation of this mechanism is to ask each seller to put down an amount of cash in the cause of avoiding or promoting a sale at an announced offer price; whichever pile is larger in aggregate wins.

\(^10\)Our approach thus naturally extends to (the constant marginal cost version of) Cournot’s original collaboration problem (Subsection VI.B) and to an analogous continuous public goods setting (Subsection VI.C).

\(^11\)We believe that the public goods literature missed the approximation to Lindahl equilibrium, which our mechanism proposes because of a focus on more general continuous public goods environments with income effects. In these more general settings, our mechanisms have no obvious analog.

\(^12\)Another class of such mechanisms that is just beginning to be studied are so-called package exchanges (Milgrom, 2007). Because in these mechanisms truthfulness is not incentive compatible, it is not clear how to square their suggestions of efficiency with Mailath and Postlewaite’s theorem. However, we suspect that incentives for deceit in those mechanisms will be most acute in holdout setting and thus, in practice, they will perform in the long-run roughly like other property right preserving mechanisms.

\(^13\)Various authors have proposed decentralized systems for solving the holdout problem. However, these systems present problems of their own. For example, Kelly (2006) argues that the ability of developers to secretly purchase land for assembly is sufficient to address the problem. While secret purchases will help reduce the holdout problem for any particular assembler, they increase the difficulty in the real estate market generally, as each seller will come to suspect that her land may be part of an assembly. Thus, the loss from holdout is just spread out over many transactions as an increased Myerson-Satterthwaite distortion. Bell
superiority of Concordance mechanisms over their alternatives as mentioned above.

Our paper is a first pass at practical solutions to holdout. However, given the paucity of work on this important question, we suspect much of our contribution will be in raising the question of holdout, rather than answering it. We therefore hope that future research will lead to superior mechanisms for alleviating holdout. In Section VI, we conclude by discussing some directions such work might follow. We collect most proofs in an appendix following the main text of the paper. Additionally, on our websites, we provide a more extensive appendix discussing applications and the connections between holdout problems and Cournot’s theory, and software (designed by William Weingarten), which implements and simulates all the mechanisms described in this paper.

I An Illustrative Example

We begin with a simple example illustrating our main contribution, using the language of our land assembly application and the notation established more formally in Section II.

We suppose that \( N = 10 \) (potential) sellers \( i = 1, \ldots, 10 \) own privately-valued plots of land, with values \( v_i \) drawn independently and uniformly from the interval \([0, 10]\). A buyer \( \beta \) has private value \( b \) for the collective plot, hence sale of the community plot to the buyer is efficient if and only if \( b > V \equiv \sum_i v_i \). Since the sellers’ values \( v_i \) and the buyer’s value \( b \) are all private, a bargaining challenge arises: how are the parties to identify and reach an efficient outcome?

Naïve self-assessment mechanisms for this problem ask sellers to simply name reserve prices \( r_i \), with the buyer acquiring the plots at those prices if and only if \( b \geq \sum_i r_i \). But these mechanisms have clear incentive problems—the seller is asked to name the price she

\cite{Parchomovsky2007} and \cite{PlassmannTideman2009} have proposed using property self-assessment to elicit truthful revelation of values. However, this mechanism is decentralized and by the extension of the Green and Laffont (1977) theorem we prove in the appendix, any such mechanism, if incentive compatible, must be payoff-equivalent to a Groves mechanism. \cite{Shoup2008} proposes regulatory subsidies for land assembly, an approach with limited applicability (and potential incentives for misuse) because it is not self-financing.
will be paid, hence she is incentivized to holdout by reporting \( r_i > v_i \).

One solution to the problem of value-shading is for \( \beta \) to make an identical take-it-or-leave-it offer to each seller, acquiring the plots if and only if all sellers accept. But this approach is ineffective, as the probability of trade via this method quickly goes to 0. To illustrate this, we suppose that the buyer makes an offer of \( 8 \)—four-fifths of the maximum possible seller value—to each seller. Sale will occur only in the probability

\[
\left( \frac{8}{10} \right)^N = \left( \frac{8}{10} \right)^{10} < .11
\]

(1)
event that no seller has value above 8. We see that although there is no value-shading, the buyer again faces a “holdout” problem: with high probability, some seller will refuse sale unless offered more than 8.\(^{14}\)

A natural solution to holdout is eminent domain: \( \beta \), with government assistance, takes the plots and compensates the sellers according to the standard of (exogenously determined) “just compensation.” To do this, the government hires an assessor (such as a real estate agent), who uses publicly-observable, objective information to determine a total value \( o \) for the community plot and the relative ownership shares \( s_i \) of the sellers. For example, suppose that these assessments are based only upon total land area: \( s_i \) is set equal to \( \frac{\ell_i}{L} \), where \( \ell_i \) is the amount of land in the plot of seller \( i \) and \( L = \sum_i \ell_i \), and \( o = L \cdot m \), where \( m \) is the market value per unit of land. With this rule and market value \( m = \frac{1}{50} \), the land areas \( \ell_i \) specified in (2) give rise to following shares \( s_i \) and seller compensation levels:

\[
\begin{array}{cccccccccc}
  i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  \ell_i & 55 & 40 & 60 & 80 & 100 & 120 & 140 & 160 & 180 & 165 \\
  s_i & \frac{1}{20} & \frac{2}{55} & \frac{3}{55} & \frac{4}{55} & \frac{5}{55} & \frac{6}{55} & \frac{7}{55} & \frac{8}{55} & \frac{9}{55} & \frac{3}{20} \\
  \text{Compensation} & 0.55 & 0.40 & 0.60 & 0.80 & 1.00 & 1.20 & 1.40 & 1.60 & 1.80 & 1.65
\end{array}
\]

(2)

\(^{14}\) It is clear from the form of (1) that this holdout problem grows quickly with \( N \). Indeed, for \( N = 20 \), the probability of trade is already less than .02.
Eminent domain guarantees that all efficient trades occur—any buyer with value \( b > 22 \) will be willing to buy the land at price \( 22 = 1100 \cdot \frac{1}{50} = L \cdot m \). Unfortunately, since it detaches the sale decision from the sellers’ values, eminent domain allows inefficient trade. To see this, suppose that \( v_i = i \) for each \( i \), so that \( V = 55 \). In this case, following the entry of a buyer with value \( b = 30 \), the land might be taken as described in the previous paragraph. This trade is inefficient since \( b = 30 < 55 = V \). Moreover, in receiving the compensation specified by (2), all sellers are drastically expropriated: the total value of these sellers is \( \sum_{i=1}^{10} v_i = \sum_{i=2}^{10} i = 55 \), and between them they receive only 22, with seller 10 receiving less than a sixth of his value.

Nonetheless, the basic idea of eminent domain—that an individual’s share of compensation is independent of her behavior—is sound. Our Concordance mechanisms build upon this principle, using \( s_i o \) as baseline compensation: an offer \( o \) is obtained from the buyer \( \beta \), and each seller \( i \) receives \( s_i o \) whenever trade occurs. But unlike eminent domain, Concordance mechanisms link the sale decision directly to reserve values \( r_i \) reported by the sellers: trade occurs if and only if the offer exceeds the collective reserve, \( o \geq R \equiv \sum_i r_i \).

If the shares are perfectly assessed, that is, if \( s_i = \frac{v_i}{V} \) for each \( i \), then this structure is all that is required to solve holdout. Sellers cannot affect the aggregate sales price \( o \), but only whether it is accepted and each seller \( i \) favors a sale if and only if \( \frac{v_i}{V} o = s_i o \geq v_i \), that is \( o \geq V \). Thus, it is in the best interest of \( i \) to truthfully reveal her value, \( v_i \). But if shares are imperfectly assessed, sellers with lower (higher) than average shares may try to prevent (encourage) a sale inefficiently by over-(under-)stating their values. A Concordance mechanism is thus obtained by combining the Concordance compensation scheme (\( s_i o \) in case of sale) and decision rule (sale if and only \( o \geq R \)) with some tax designed to discourage such dishonestly. Such a tax can be based on computing average externalities or revealing them through an equilibrium market process analogous to an all-pay or first-price auction. The simplest procedure to illustrate, however, is based on using other sellers’ reports to compute externalities—the VCG mechanism.
In particular, one can say a seller is *pivotal in the sale decision* if the community would have supported a different decision were \( i \) replaced by a seller indifferent to the sale \((v_i = s_i o)\). Following VCG, our Straightforward Concordance (SC) mechanism forces all pivotal sellers to pay for the externality they cause, and refunds some of these tax revenues to the sellers using a rule due to Cavallo (2006). As with any VCG mechanism, it is a dominant optimal for sellers to report their values as they are forced to internalize any ill effects their report has on others. Therefore, it is also optimal for the buyer \( \beta \) to make the monopsonist-optimal offer. With \( b = 78 \) and the specific distributional assumptions already imposed, this optimal offer is close to 56; we assume \( o = 56 \) for expositional simplicity.

We next consider what happens when shares do not perfectly represent relative values, as in (2): \( s_1 = \frac{1}{20} \) (which is too high) and \( s_{10} = \frac{3}{20} \) (which is too low). Now, in case of sale, seller 1 (10) will receive an unfairly large (small) fraction of the offer. Nonetheless, since a Groves taxation scheme is used, truthful reporting still occurs in equilibrium, and hence sale occurs (since 56 > 55). Seller 1 is pivotal, since \( \sum_{j \neq 1} r_j - s_1 \approx 56.84 > 56 \), and pays a tax equal to her externality, \( (1 - \frac{1}{20})|56.84 - 56| \approx .8 \). (Rounded) SC compensation levels are given by

\[
\begin{array}{c|cccccccccc}
  i  & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  \hline
  \text{Payoff} & 2.0 & 2.0 & 3.1 & 4.1 & 5.1 & 6.1 & 7.1 & 8.2 & 9.2 & 8.4 \\
\end{array}
\]

Only the property rights of seller 10 are violated, and even those rights are almost preserved.\(^{15}\) Note that each non-pivotal seller \( i = 2, \ldots, 10 \) receives at least

\[
\frac{s_i \sum_{j \neq i} v_j}{1 - s_i}, \tag{3}
\]

the (share-weighted) average (implied) value of sellers \( j \neq i \) for the plot of seller \( i \).\(^{16}\) Each non-pivotal seller is paid according to the “community consensus on the severity of the harm inflicted,” which some legal scholars (Ellickson, 1973) believe to be the appropriate founda-

\(^{15}\)Compare to the case of eminent domain pictured in (2), in which seller 10 received only 3.3!\(^{16}\)This value (3) equals seller \( i \)'s share of the collective reserve that would have obtained were \( i \) absent.
tion for “just compensation.” Any seller $i$ can opt to exert no influence on the collective decision by stating $r_i = s_i o$. Therefore this outcome provides a lower bound on seller payoffs in Concordance mechanisms. As the assessment of the shares $s_i$ becomes more accurate (i.e. as $s_i \rightarrow \frac{v_i}{v}$ for all $i$) (3) converges to $v_i$, hence the payoff guarantee of Concordance mechanisms is an approximation to individual property rights protection. The quality of the approximation is mediated by the noise level of the share assessment. To see this, consider the following table, which presents the bound (3) with different noise levels (measured by $\sigma$) added into the share assessment process:

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>$\sigma = \frac{1}{12}$</td>
<td>0.7</td>
<td>2.2</td>
<td>2.9</td>
<td>3.9</td>
<td>4.7</td>
<td>5.8</td>
<td>7.6</td>
<td>7.9</td>
<td>9.5</td>
<td>10.0</td>
</tr>
<tr>
<td>$\sigma = \frac{1}{3}$</td>
<td>0.0</td>
<td>2.1</td>
<td>3.2</td>
<td>3.6</td>
<td>5.2</td>
<td>5.4</td>
<td>7.7</td>
<td>7.4</td>
<td>9.2</td>
<td>11.4</td>
</tr>
</tbody>
</table>

In (4), as the noise parameter $\sigma$ decreases, the variance of the residual difference between $v_i$ and the property rights guarantee of seller $i$ (represented by the $(\sigma,i)$ cell in (4)) decreases; for $\sigma = 0, \frac{1}{12}, \frac{1}{3}$, these variances are 0, .1, and .5, respectively.\(^{17}\)

In our example, the total community payoff is $56 - .8 = 55.2$. As with any other VCG implementation, the SC mechanism is not budget-balanced—some of the buyer’s offer $o = 56$ is lost in taxes. Nonetheless, $55.2 > 55 = V$, hence the community together receives at least its aggregate value, $V$.\(^{18}\) Since sale in a Concordance mechanism occurs if and only if $o \geq R$, this community property rights preservation can be guaranteed in general. Indeed, in SC the community receives at least $V$ in sale if each seller $i$ reports $r_i = s_i V$ (whence $R = \sum_i r_i = \sum_i s_i V = V \sum_i s_i = V$) since this reporting strategy never generates SC taxes. Unfortunately, in SC this behavior does not arise under truthful reporting unless shares are

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\(^{17}\)This table was generated (by William Weingarten), by generating values $\tilde{v}_i = v_i + \epsilon$, where $\epsilon$ is uniform with variance $\sigma$, and then taking $s_i = \tilde{v}_i / \left( \sum_j \tilde{v}_j \right)$ in (3).

\(^{18}\)In our example, the preservation of community property rights in equilibrium is a consequence of our distribution and value assumptions.
perfectly assessed. Our alternative mechanisms (such as BNC mechanism of Subsubsection V.B.1) are budget-balanced, and therefore ensure community property rights when sellers are truthful.

As we see in our example, sale occurs in SC if and only if the monopsonist-optimal offer for the whole plot exceeds the community value \( o \geq R = V \). Thus, the sale decision is *bilaterally efficient*, i.e. it mimics a bilateral bargain between the buyer \( \beta \) and a single seller—the community—with value \( V \). Moreover, since the seller values \( v_i \) are drawn independently, (by the law of large numbers) the uncertainty regarding the efficiency of trade decreases as \( N \) grows large. When \( N = 10, b = 78, \) and \( o = 56 \) as above, the probability that \( o < V < b \) (trade is efficient, but the offer is too low) is only .26—inefficiency obtains in just over a quarter of cases. As \( N \) grows, the probability of inefficiency drops sharply: for \( N = 20 \), it is .04, and for \( N = 30 \) it is \( 1.2 \cdot 10^{-6} \). It follows that SC is *asymptotically efficient*—SC becomes fully efficient in the limit as \( N \to \infty \).

The main sections of the remainder of this paper (Sections II, III, and V) show that the attractive property rights and efficiency guarantees we have illustrated here hold for Concordance mechanisms more broadly.

## II The Model

For concreteness, we will discuss our model in terms of a land assembly example: there are \( N > 0 \) (potential) sellers \( i \), each of whom owns a piece of a plot of land, valuing it at \( v_i \) dollars.\(^{20}\) There is a single (potential) buyer \( \beta \), who is only interested in buying the entire plot of land, which she values at \( b \) dollars.\(^{21}\) We assume that \( v \equiv (v_1, \ldots, v_N) \) is drawn from \( [\underline{v}, \overline{v}]^N \) according to a smooth, full support joint probability density function \( g \), and that \( b \) is independent of \( v \). We let \( V \equiv \sum_i v_i \), let seller \( i \)'s share \( s_i(\tilde{V}) \equiv E \left[ \frac{v_i}{\tilde{V}} | V = \tilde{V} \right] \), and write

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19 This is a consequence of the fact that when the efficiency of trade is certain, bilateral bargaining is fully efficient.

20 We assume sellers value only their own plots of land.

21 We will typically suppress the notation \( \beta \) except in mathematical expressions.
\( s \equiv (s_1, \ldots, s_N) \). We assume \( s_i \left( \tilde{V} \right) \) is constant in \( \tilde{V} \) at \( s_i \).

Section IV discusses how shares can be ascertained in practice for particular applications. Transactions are structured by a mechanism (Hurwicz, 1973) consisting of a collection of offers \( O \subseteq \mathbb{R} \) specifying actions that buyers can take, reserve values \( R \subseteq \mathbb{R} \) specifying actions that sellers can take, a purchase rule \( P : B \times R^N \rightarrow \{0, 1\} \) specifying the actions of buyers and sellers following which a transfer of the plot takes place, and a transfer rule \( T : B \times R^N \rightarrow \mathbb{R}^{N+1} \) specifying transfers to or from buyers and sellers that follow actions.\(^{22}\)

The mechanism may also have a pair of suggested strategies \((o^*, r^*)\) that, while not technically part of the mechanism per se, help in structuring thought about the mechanism.\(^{23}\) Here, \( r^* : [\underline{v}, \overline{v}] \rightarrow R \) is a suggested seller reserve function, and \( o^* \) is a suggested buyer offer function.\(^{24,25}\)

Throughout, we use the stylistic convention that offers and reserve values are reported simultaneously. However, this simultaneity is not strictly necessary in any of our mechanisms. Indeed, in all our mechanisms the buyer pays her full offer in the case of sale. This offer may therefore be made and revealed to sellers before sellers’ reserves are reported.

There are a number of properties that one might want a mechanism to satisfy. A basic one is that the mechanism not require external subsidies.

\(^{22}\)We write \( T_i(o, r) \) for the transfer to the buyer and write \( T_i(o, r) \) for the transfer to the seller \( i \).

\(^{23}\)We use this (nonstandard) notational convention to highlight the fact that, just as much of market design analyzes mechanisms’ outcomes under ‘“suggested”’ truthful behavior (e.g. Roth (2008), Budish (2008)), we conduct our overarching analysis with respect to suggested strategies. For each mechanism we present, we discuss the degree to which the suggested strategies are equilibrium behavior.

\(^{24}\)In principle, some mechanisms might recommend different reserves to different sellers depending on their identity (or share) and not merely based on their valuation. However, any such mechanism may easily be transformed, by the revelation principle (Gibbard, 1973), into one where the suggested strategy is truthful revelation. Thus there is no loss of generality in our approach of economizing on notation.

\(^{25}\)For a simple example of this notation, we give the formalization of the \( \frac{1}{2} \)-plurality mechanism as the mechanism in which the buyer makes a take-it-or-leave-it offer that is revealed to the sellers, who vote on whether to accept the offer (with share-majority ruling). If no sale is made, no money changes hands; if a sale is made, each seller receives her share of the buyer’s offer. This mechanism’s offer and reserve domains are \( O = \mathbb{R}_+ \) and \( R = \mathbb{R}_+ \). The purchase rule is given by \( P(o, r) = 1_{\sum_i s_i \geq \overline{v}} \) where \( 1_\Xi \) is the indicator function for \( \Xi \). The transfer rule is \( T(o, r) = (-1, s) P(o, r) o \): conditional on a sale, the buyer pays her bid \( b \) to the sellers in proportion to their shares and in the event of no sale, no money changes hands. Sellers are to reveal truthfully, \( r^*(v) \equiv v \). Finally, \( o^*(b) \) is the monopsonist optimal price facing supply \( \tilde{G}^2 \), the cdf of the minimum successful offer.
Definition 1. $\mathcal{M}$ is self-financing if $T_\beta + \sum_i T_i \leq 0$. A self-financing mechanism $\mathcal{M}$ is budget-balanced if $T_\beta + \sum_i T_i = 0$.

Mechanisms that are not self-financing may be open to fraudulent exploitation and are therefore virtually never used in market design (Myerson and Satterthwaite, 1981) or industrial organization (Tirole, 1988). Market designers may also want to require the stronger condition that no money be paid to the social planner.

Sellers in the applications we describe in Section IV are often inexperienced, under-resourced or otherwise ill-equipped to make complex calculations. It is therefore useful for them to not have to think overly carefully about how to participate in the mechanism. One way to achieve this is for the strategy suggested to the sellers to always be in the sellers’ interests.

Definition 2. A mechanism $\mathcal{M}$ is straightforward for sellers (buyers) if the suggested seller (buyer) strategy $r^\star$ ($o^\star$) is dominant.

Extending the results of Green and Laffont (1977) shows that the set of efficient straightforward mechanisms is highly restricted. A weaker requirement is (Bayes-Nash) implementability: if all participants but one follow suggested strategies, then the last participant finds it in her interest to follow her suggested strategy.\footnote{Implementability requires participants to anticipate others’ actions, so it will typically depend on participants’ beliefs. We follow the Harsanyi doctrine in modeling participants’ beliefs as derived from the common prior belief held by the social planner, and updated according to whatever information they hold.}

Definition 3. A mechanism is (Bayes-Nash) implementable if the suggested strategies form a Bayes-Nash equilibrium of the game defined by the mechanism.

A final notion of incentive compatibility is not worth formally defining as it is used only informally in the sequel, but is Nonetheless, of interest. In land assembly, shares are often determined by the buyer or someone with the buyer’s interest primarily in mind, such as a local government. It would be desirable if that authority have no incentive to distort the

\footnote{See Weyl and Tirole (2010) for a more general discussion of reasons why subsidies are undesirable in mechanism design.}
shares. We will say that a mechanism is share incentive compatible if a buyer could not gain by misreporting shares, given suggested strategies.

A natural goal is allocative efficiency. A fully efficient mechanism achieves this goal perfectly when it implements trade exactly when trade is beneficial.

**Definition 4.** A mechanism is fully efficient if, for all \((b,v) \in B \times \mathcal{R}^N\), we have

\[
P(o^*[b], r^*[v]) = 1_{b \geq \sum_i v_i}.
\]

As discussed above, protecting property rights (nearly) always involves sacrificing some efficiency. A simple alternative to full efficiency is to guarantee that outcomes are no worse than under bilateral trade.

**Definition 5.** The efficiency of a mechanism \(e(\mathcal{M}) \equiv \frac{E[(b-V)P(o^*[b], r^*[v])]}{E[(b-V)1_{b>V}]}\), the fraction of total possible gains from trade realized. A mechanism \(\mathcal{M}\) is bilateral efficient relative to another mechanism \(\mathcal{M}'\) with \(N = 1\) if the efficiency of \(e(\mathcal{M}) \geq e(\mathcal{M}')\) when the (\(b\)-conditional) distribution of the single seller valuation under \(\mathcal{M}'\) is the same as that of \(V\) under \(\mathcal{M}\).

Most mechanisms we discuss have natural analogs for collections of sellers of any size. We can therefore think of these mechanisms as forming a series \(\{\mathcal{M}_i\}_{i=1}^\infty\) and analyze whether a mechanism becomes more or less efficient as the number of sellers grows large.

**Definition 6.** A series of mechanisms \(\{\mathcal{M}_i\}_{i=1}^\infty\) is asymptotically efficient relative to a series of joint probability density functions if \(\lim_{i \to \infty} e(\mathcal{M}_i) = 1\) under the induced measures.

Another goal is preservation of property rights, that no seller should be forced to sell below her value. A stricter standard is that her suggested action have this property.

**Definition 7.** A mechanism preserves individual property rights if, for all \(v_i \in [\underline{v}, \overline{v}]\), there exists \(r \in \mathcal{R}\) such that

\[
T_i(o^*[b], r, r^*[v_{-i}]) \geq v_i P(o^*[b], r, r^*[v_{-i}])
\]
for each $i = 1, \ldots, N$ and $(b,v_{-i}) \in \mathcal{B} \times \mathbb{R}^{N-1}$. A mechanism strictly preserves individual property rights if $r^*$ has this property.

The many philosophical, legal and economic reasons why protecting property rights may be attractive are discussed more extensively in Section IV and in the online appendix. However, the results surveyed in Section I show that preserving individual property rights is inconsistent with social efficiency, hence practical mechanisms for solving the holdout problem must abrogate some property rights. Nonetheless, a primary goal of our work is to place concrete and tolerable bounds on the maximum individual property rights loss needed to ensure efficiency. One goal is to protect the property rights of the community as a whole; that is, the community should never be forced to sell the land for less than the aggregate community value.

**Definition 8.** A mechanism preserves collective property rights if for all $v \in [v, \bar{v}]^N$, there exists $r \in \mathcal{R}^N$ such that $\sum_i T_i(o,r) \geq V \cdot P(o,r)$ for each $(b,v) \in \mathcal{B} \times \mathcal{R}^N$. This condition holds strictly if $r^*$ satisfies this property.

As discussed more extensively in Section IV and the separate appendix, collective property rights may both satisfy legal norms in some settings and help preserve collective investment incentives, useful in many settings (such as corporate acquisitions) where investments are primarily collective. Nonetheless, community property rights may be cold comfort to expropriated individuals within the community. While precisely preserving individual property rights may be impossible, a more modest goal is that each seller receive a “fair” share of the collectively acceptable compensation.

**Definition 9.** A mechanism preserves approximate individual property rights if, for all $i = 1, \ldots, N$, and $(b,v) \in \mathcal{B} \times [v, \bar{v}]^N$, there exists $r \in \mathcal{R}$ such that

$$T_i(o^*[b], r, r^*[v_{-i}]) \geq \frac{s_i \sum_{j \neq i} v_j}{1 - s_i} P(o^*[b], r, r^*[v_{-i}]).$$
Preserving approximate individual property rights seems likely to maintain appropriate incentives for individual investments whose benefits are observable by the social planner, though Again,a formal analysis, omitted here, would be helpful to establish this. Furthermore, (see the separate appendix) it satisfies, perhaps even more precisely than preservation of individual property rights, some of the philosophical and legal criteria on which constitutional requirements of “just compensation” for deprivation of property are founded.

III The Concordance Principle

Our basic approach, of which all of the mechanisms we propose can be seen as applications, is inspired by Cournot’s solution to the collaboration problem. Cournot argued that the collaborating firms should merge so as to fairly share in—and hence internalize—each others’ profits. We see this suggestion, applied to holdout, as consisting of two parts:

1. Sellers should divide profits from a sale according to a pre-specified formula, just as a merger divides stock in the conglomerate among the shareholders of the merging firms.

2. Sellers should be incentivized to share information by paying for externalities caused by moving the group decision towards her preference, just as divisions of a firm (Groves and Loeb, 1979) are incentivized to communicate with headquarters.

The Concordance principle, so-named in honor of Cournot’s theory, implements these ideas. First, decisions are made to maximize the community’s aggregate welfare and the offer is assigned according to exogenous shares. Second, any land-owner who directly affects the sale decision, relative to the outcome if that individual’s share were evenly distributed over the rest of the community, is required to pay for her impact. This procedure is detailed informally in the box above and formally in the following definition.

28Lehavi and Licht (2007) suggest the notion of a “merger” as well in abstract terms, but without any explicit decision procedure, which we view as inseparable from the Concordance Principle.
The Concordance Principle

1. Sellers are asked to report their values truthfully and buyers are asked to make the monopsonist-optimal offer to the aggregate seller.

2. The buyer’s offer is accepted when it exceeds the total reported reserve.

3. Each seller has the option to exert no influence in which case the share-scaled-up reserve of all other sellers determines whether a sale occurs. If no seller exerts influence the sale proceeds.

4. Sellers exerting no influence, and sellers exerting influence alone, receive at least their share of the offer if a sale occurs and never pay anything.

5. In order to encourage truthful reporting, sellers exerting influence may be required to pay a Pigouvian tax.

Definition 10. A mechanism \( \mathcal{M} \) satisfies the Concordance principle if

\[
\left[ r_i = s_i o \right] \lor \left[ r_j = s_j o \ (\forall j \neq i) \right] \implies T_i(o, r) \geq s_i o P(o, r),
\]

\[ r^*(v) \equiv v \text{ and } o^*(b) \equiv \arg\max_o (b - o) G(o). \]

We henceforth call any mechanism satisfying the Concordance principle a Concordance mechanism and refer to the set of all such mechanism by \( \mathfrak{C} \). The most distinctively attractive properties of these mechanisms (bilateral and asymptotic efficiency, collective and approximate individual property rights) follow directly from the Concordance principle, as we show in Theorems 1 and 2.

Theorem 1. Every \( \mathcal{M} \in \mathfrak{C} \) is bilaterally efficient relative to its own bilateral form (a take-it-or-leave-it offer to a single seller). Furthermore, a sequence of mechanisms \( \{\mathcal{M}^n\}_{n=1}^\infty \) with \( \mathcal{M}^n \in \mathfrak{C} \) for all \( n \) is asymptotically efficient given share vectors \( \{s^n\}_{n=1}^\infty \) if

1. there exists an \( M > 0 \) such that for all \( i \) and \( n \), \( ns^n_i < M \),
2. \( \left\{ \frac{v_i^n}{s_i^n} \right\}_{i=1}^n \) are drawn independently and identically across \( n \) and \( i \) from some finite-support distribution, and \( b \) is drawn independently and identically across \( n \).

Thus Concordance mechanisms solve the holdout problem: their efficiency does not deteriorate below that achieved in bilateral trade as the number of sellers grows large. In fact, these mechanisms transform pessimistic scenario of collaboration into the optimistic one of competition: provided seller values are somewhat independent, Concordance mechanisms become fully efficient as the number of sellers grows large.\(^{29}\)

Concordance mechanisms use the aggregate seller value as an offer threshold. This is the same rule as if the community were a single seller, implying bilateral efficiency. Furthermore, if the ratio of seller values to shares are drawn i.i.d. and shares are not too concentrated, then as the number of sellers increases, by the law of large numbers, asymmetric information about aggregate valuation dwindles and the Cournot-Myerson-Satterthwaite distortion vanishes.\(^{30}\)

Concordance mechanisms also satisfy both of our second-best property rights guarantees. They use the same decision rule for accepting offers as the whole community would, so they preserve collective property rights as long as the community can avoid tax payments.\(^{31}\) Each seller, if she exerts no influence, receives her share of the offer, which is—if accepted—at least \( s_i \sum_{j \neq i} r_j \); consequently, her approximate property rights are never violated when other agents play the suggested (truthful) strategy.

**Theorem 2.** All \( \mathcal{M} \in \mathcal{C} \) preserve collective and approximate individual property rights.

All the additional details distinguishing Concordance mechanisms relate to how they use taxes to encourage truthfulness. The various techniques (VCG, expected externality, all-pay,

\(^{29}\)It is clear from the proof of Theorem 1 that full independence of seller values is not needed; weak mixing conditions would suffice. Indeed, the independence assumption is used in the proof only for a variance bound and an application of Chebyshev’s inequality.

\(^{30}\)As with bundling of sufficiently many independently valued goods (Armstrong, 1999), the buyer eventually appropriates all (potential) gains from trade. If aggregate uncertainty about seller values persists in the limit, so will distortions.

\(^{31}\)Note that all tax payments can be avoided in this case if, for example, one seller submits her share of the community valuation and every other seller chooses not to exert influence.
first-price) we use to encourage truthful revelation induce trade-offs along dimensions familiar to auction theorists: straightforwardness, budget-balance and attention to the “Wilson (1987) doctrine” (avoidance of strong dependence on beliefs). The Concordance mechanisms we propose are, with some small modifications, simple combinations of the Concordance principle with these auction-theoretic enforcements. Thus the Concordance principle is not a substitute for standard auction approaches; rather it is a complement, providing a means of making standard auction theory relevant and appealing for the holdout problem.

IV Applications

The abbreviated discussion of applications presented here is extended in our online appendix.

IV.A Land assembly

Large plots of contiguous land valued greatly in their entirety are often divided into pieces owned by disparate individuals. To help alleviate the holdout problem that (especially public) developers face in assembling land (Posner, 2005), the policy of “eminent domain” in the Fifth Amendment to the United States (U.S.) Constitution allows the government to take “private property ... for public use,” but only after “just compensation” has been paid. This procedure typically involves the (local) government asking for an assessment of land values by a real estate expert. Given limited legal recourse for takees following recent Supreme Court decisions (Kelo v. City of New London, Connecticut, 2005), this compensation is far below the minimum price at which residents would sell. States have reacted by severely curtailing the use of eminent domain (Castle Coalition, 2009; Morton, 2006), restoring the holdout problem.32

Land assembly has played important roles in societal development. Hoffman (1988)...

32Inability to resolve holdout problems is common in developing countries, but some have taken a middle path, requiring the consent of (only) some fraction of sellers as recently proposed for the U.S. by Heller and Hills (2008).
argues that Britain beat France to industrialization largely because, unlike France, it required only four-fifths of owners to consent to assemblies. The Mexican ejido system of collective land ownership requires the consent of all community members for any land to be sold, leading to a “national agricultural system dominated by uneconomically small...farms” (Schmidt and Gruben, 1992). By contrast, in the immediate post-war period a Japanese system of partial-consent land assembly sped development (Minerbi, 1986); its decline since then is blamed for much sprawl (Sorensen, 1999) and the eventual Narita airport debacle (Shimizu, 2005).33

Moreover, the setting of land assembly is closely linked to the concepts we introduced in Section II. Individuals’ shares of the total assessed value of to-be-assembled land can be used without directly basing compensation upon assessments. Collective property rights have a natural place in systems (such as México) where land is legally owned jointly by a community. Approximate individual property rights are a natural formalization of the “community consensus on the severity of the harm inflicted” that some legal scholars (Ellickson, 1973) believe to be the appropriate foundation for compensation.

IV.B Corporate acquisitions

When one individual or corporation seeks a controlling share in a public firm, most countries require that it makes a bid for all shares (Kirchmaier et al., 2009).34 Because individuals have heterogeneous risk-aversion and belief-driven infra-marginal utility from investing in the to-be acquired firm, it would be nearly impossible for a prospective buyer to voluntarily purchase all shares. Thus to allow acquisitions to take place, nearly every jurisdiction allows consent by some super-majority of share-holders to squeeze-out (Croft and Donker, 2006) or

33 Additionally, the land assembly necessary to create a crime-free neighborhood is widely cited as one of the important reasons why extensive low-value slums continue to occupy some of the world’s most beautiful urban real estate on the hills overlooking Rio de Janeiro. Massive land assembly problems (FERC, 2005) confront builders of the energy infrastructure (pipelines and transmission lines) that will be required in coming years (Chupka et al., 2008). Efficient collective decisions about the use of land are widely thought to be reason for the dominance of the condominium form of property management (West and Morris, 2003; Heller, 2008).

34 These regulations are designed to protect minority shareholders’ interests in the case of take-overs by other firms whose interests do not concord with strict divisional profit maximization and to help ameliorate free-riding on corporate efficiency improvements (Grossman and Hart, 1980) by corporate “raiders.”
over-rule (Armour and Skeel, 2007) the remaining holdouts.

The criteria we defined seem compelling here. Collective property rights are appropriate given that protecting collective, rather than individual, investment incentives is paramount.\textsuperscript{35} Guaranteeing each individual a share-weighted fraction of the collective settlement corresponds to paying that individual's share of the acquisition price.

\textbf{IV.C Other examples}

Other examples abound. Approaches to land assembly can be applied (as discussed in Subsection V.A) as easily to land disassembly as to land assembly, as the same collective action problem arises. This offers an alternative to the expropriation usually involved in land reform. Rules in most countries (La Porta et al., 1998) require the consent of a supermajority of creditors to a debt renegotiation outside of bankruptcy, with thresholds differing across countries. Following Federal Communications Commission (FCC) auctions, radio spectrum has become fragmented, inhibiting efficient high-speed wireless internet (Hazlett, 2005); re-assembling this spectrum is a top priority of the FCC. Investors commonly assemble pools of complementary patented innovations and license them jointly (Lerner and Tirole, 2004), but difficulty forming pools can be a drag on innovation (Heller and Eisenberg, 1998). Class action legal settlements are often plagued by holdouts (Rob, 1989). A collector trying to assemble the pieces of a fragmented triptych or a closely connected group of artworks faces holdouts among the current owners of individual pieces.\textsuperscript{36}

\textbf{IV.D How big is the holdout problem?}

In the U.S. alone, there are nearly 6000 active takings (Berliner, 2006) per year; these likely represent only a small fraction of all land assembly in the U.S.. Supposing an average stake of $10 million, these represent $60 billion annually; similar guesses for México and Brazil

\textsuperscript{35}It is hard to imagine shareholders making substantial \textit{individual} investments in the value of the firm.

\textsuperscript{36}Heller (2008) surveys a variety of other examples, from post-Communist property transitions in eastern Europe to share-cropping relations in the post-Bellum South.
indicate together at least $20 billion dollars annually. Thus global land assembly activity is likely on the order of hundreds of billions of dollars each year. According to Dealogic, corporate acquisitions amounted to $972 billion or 5.5% of global GDP in the first quarter of 2008 (Twaronite, 2009) alone. When the economy is weak, reduced acquisitions are compensated by debt settlements; in 2008, according to BankruptcyData.com, the assets of U.S. firms filing for bankruptcy amounted to more than a trillion dollars. Aggregating these and the other activities mentioned above gives a ballpark estimate of many trillions of dollars for the annual volume of transactions subject to holdout problems.

Supposing an average of 20% potential gains from trade (consistent with a modest 10% monopoly mark-up under linear demand), and assuming that a very modest one quarter of these are lost to deadweight from holdout, this amounts to 5% of transaction volume.\(^{37}\) A high 5% real interest rate would roughly indicate that the discounted NPV of social gains from an efficient mechanism for holdout is on the order of many trillions of dollars or double digit percents of global annual GDP.\(^{38}\)

V Concordance Mechanisms

V.A Straightforward Concordance (SC)

The simplest implementation of the Concordance principle uses the mechanism of Vickrey, Clarke and Groves to enforce truthful revelation of values. This mechanism incentivizes truthful revelation through Pigouvian externality taxes assessed on the base of other sellers’ valuation reports.

Straightforward Concordance, outlined informally in the box above, is formally defined

\(^{37}\)This is linear demand monopoly deadweight loss, which is much smaller than for other demand functions. Furthermore, monopoly deadweight loss is very modest compared to what one would expect from holdout.

\(^{38}\)Thus we believe our estimates indicate that holdout is not just a problem of theory, but a pressing social challenge of practical importance at least as great as the problems of auctions (Klemperer, 2000; Milgrom, 2004), matching (Roth, 2008), assignment (Abdulkadiroğlu and Sönmez, 1998) and combinatorial assignment (Budish, 2008).
Straightforward Concordance

1. SC is a Concordance mechanism with taxes on sellers who are pivotal in the sense that $R$ and $R_i$ are on different sides of $o$.

2. Pivotal sellers $i$ pay a tax equal to the harm caused: $(1 - s_i)|o - R_i|.$

3. A refund is paid to each seller $i$ of her share of the minimal VCG surplus independent of the reservation seller $i$ announces.

by $\mathcal{B} = \mathcal{R} = \mathbb{R}_{++}, P(o, r) = 1_{o \geq R}, T_\beta(o, r) = -oP(o, r),$

$$T_i(o, r) = s_i o P(o, r) - 1_{(R_i-o)(o-R_i)<0} (1 - s_i)|o - R_i| + s_i \min_{\hat{r}_i} \sum_{j=1}^{N} \left( 1_{(\hat{R}_j-o)(\hat{R}_j-o)} (1 - s_j)|o - \hat{R}_j| \right).$$

Suggested strategies are monopsonist-optimal and truthful for the buyer and sellers respectively, as with any Concordance mechanism. The crucial advantage of SC over other implementations is that it is straightforward for sellers, thus its name. This implies that stated benefits associated with the Concordance principle apply not only when sellers blindly “do what they are told” to by the mechanism, but also if they act rationally in their own interests.

**Proposition 1.** $\mathcal{M}^*$ is self-financing, straightforward for sellers and implementable.

In the Appendix, we develop the class of “Groves Holdout” mechanisms, the natural extension of SC to accommodate alternative Groves payment rules. An extension of the main result of Green and Laffont (1977) to the (binary) case of holdout shows that only these mechanisms are simultaneously straightforward for sellers, Pareto-optimal among sellers, and bilaterally efficient.\(^{39}\) This characterization of the Groves Holdout class, combined with bounds derived from the Concordance principle pin down SC uniquely: any straightforward Concordance mechanism is SC up to changes in the strictly positive refund.\(^{40}\)

\(^{39}\)A few innovations are required in this argument, since the argument of Green and Laffont (1977) relies upon the existence of at least three outcomes, while our setting has only two.

\(^{40}\)In fact, we can prove a variant on this result (Proposition 9) that pins down the pre-refund structure of
Proposition 2. Any straightforward-for-sellers Concordance mechanism \( \mathcal{M} \) is isomorphic to a mechanism of the form \( \mathcal{B} = \mathcal{R} = \mathbb{R}_{++}, P(o,r) = 1_{o \geq R}, T_\beta(o,r) = -oP(o,r), \)

\[
T_i(o,r) = s_i o P(o,r) - \left[ 1_{(R_i - o)(R - o) < 0} (1 - s_i) |o - R_i| + h_i^+(o,r_{-i}) \right], 
\]

with \( h_i^+(o,r_{-i}) \geq 0 \) for all \( (o,r_{-i}) \in O \times \mathcal{R}^{N-1} \).

Furthermore, as shown in Proposition 8 in Subappendix B.3, our refund system uses the approach of Cavallo (2006) to return the maximal revenues to sellers while maintaining self-financing and without favoring any seller ex ante.\textsuperscript{41} Thus, SC is the maximally refunding, self-financing, non-discriminatory straightforward Concordance mechanism. However, in practice SC may return very little revenue; when all sellers shares are identical it never returns any.\textsuperscript{42} The mechanism is also highly vulnerable to collusion or other manipulation.

In one form of collusion effective against SC, groups can avoid tax payments by share-weightedly averaging their values and each reporting their share of this average. This leads to exactly the same sales rule as when members of the community act non-cooperatively, and additionally, community property rights are strictly preserved. Since we are concerned with achieving efficiency and protecting property rights, rather than raising revenue, collusion in this fashion actually \textit{improves} outcomes. Of course, collusion among sub-groups of sellers, or imperfect collusion among all sellers, can be harmful to efficiency and property rights. The problem is well-known to be particularly severe (Ausubel and Milgrom, 2005) if, as seems likely in corporate acquisitions for example, there is a very large number of sellers and sellers can easily “de-merge,” splitting one individual into two, each with half the share, who

\textsuperscript{41}One deficiency of this approach is that no refunding occurs if seller reports are allowed to be unbounded. Seller values are bounded in practice, however, so this does not seem particularly restrictive. The optimal tightness of these bounds would trade off expressiveness of the mechanism against budget-balance and reduction of the potential gains from manipulation.

\textsuperscript{42}To see this, note that if any seller were to submit a reservation value maximally in the direction of the outcome that would occur if she exerted no influence, then no seller (if all had equal shares) would be influential. Thus the minimal tax for any seller is 0 and thus none can safely be returned any revenue.
can then express the same extreme preference, obtaining their desired outcome at no cost. Concerns about the possibility of such manipulation, as well as imperfect budget balance, are the primary motivation behind our other Concordance mechanisms.

V.B Other Concordance Mechanisms

We now discuss three alternatives to SC that, by using other standard auction concepts, sacrifice straightforwardness to improve budget balance and, potentially, reduce collusion.

V.B.1 Bayes-Nash Concordance (BNC)

Incentives for truthful reporting of valuations in the SC are maintained by forcing sellers to pay for any external harms caused by their influence. However, since a seller’s tax payments are assessed with respect to the value disclosures of other sellers, even with refunds, SC is vulnerable to collusion amongst multiple sellers. This form of collusion may be sidestepped by using only external information in assessing sellers’ taxes. If the expected externality a seller exerts based on having a particular value is known, she can simply be made to pay this as suggested by Arrow (1979) and d’Aspremont and Gérard-Varet (1979). Implementing such an “expected externality mechanism” of course requires knowledge by the mechanism administrator of the distribution of other seller valuations conditional on a particular seller valuation.\(^43\) Such knowledge would be difficult to come by in practice, likely requiring a large data set of past similar situations to estimate the distribution of valuations as in standard structural estimation of auctions. This violates the Wilson (1987) doctrine that mechanisms should not depend too heavily for their performance on the beliefs of agents.\(^44\) Nonetheless, the mechanism is of some intellectual interest and helps frame the connection between SC and the other, perhaps more practical, mechanisms we describe below.

\(^{43}\)Cremer and Riordan (1985) argue that this can also be delegated to the buyer or even one of the sellers.

\(^{44}\)Furthermore, the incentive properties of the mechanism rely heavily on sellers being risk-neutral, perhaps reasonable over the relevant range of values, but perhaps not given that bidders in auctions for small stakes typically exhibit strong risk aversion. On the other hand, by helping to smooth payments over states, BNC would also be preferred by risk-averse sellers.
Bayes-Nash Concordance

1. BNC is the Concordance mechanism with taxes equal to expected externalities, conditional on the reported valuation and the offer.
2. Sellers receive refunds equal to her share of others’ expected externalities.

We therefore assume for the purposes of this mechanism that

1. the social planner knows the distribution of \( v \) precisely, and
2. the valuations \( v_i \) are independent of one another and of \( b \), so that the distribution of other valuations does not depend on the value of any particular seller’s valuation.

In this case, we can calculate for any reported \( v_i \) and offer \( o \) the expected Pigouvian tax the seller would have to pay under SC. This is just

\[
(1 - s_i) E_{v_{-i}} \left[ |V_i - o| 1_{(V_i - o)(V - o) < 0} \mid v_i = r_i \right].
\]

It is well-known that a mechanism in which sellers pay their expected externalities will be Bayes-Nash incentive compatible. That is, if each seller is made to pay her expected externality, then if she is expects all other sellers to reveal their valuations truthfully and she is risk neutral, she will have an incentive to reveal her valuation truthfully as well.\(^45\)

This intuition leads to the Bayes-Nash Concordance (BNC) mechanism described informally in the box above and formally by \( B = \mathcal{R} = \mathbb{R}_{++}, \ P(o, r) = 1_{o \geq R}, \) and \( T_i(o, r) \) given by

\[
s_i o P(o, r) - (1 - s_i) E_{v_{-i}} \left[ |V_i - o| 1_{(V_i - o)(V - o) < 0} \mid v_i = r_i \right] - s_i \sum_{j \neq i} E_{v_{-j}} \left[ |V_j - o| 1_{(V_j - o)(V - o) < 0} \mid v_j = r_j \right].
\]

\(^45\)In fact, Williams (1999) show that any efficient Bayesian incentive compatible mechanism must have this format, a result we plan to extend in a subsequent paper to show that BNC has payoffs equivalent to those of any implementable Concordance mechanism with risk-neutral agents.
The mechanism is budget-balanced, as the total of taxes paid equal total refunds, as shown in Subappendix C.1. Again, suggested strategies follow the Concordance principle.

The fact that BNC is actually a Concordance mechanism is immediate from the mechanism’s definition once we observe that the expected externality

$$(1 - s_i)E_{v-i} \left[ |V_i - o|1_{(V_i-o)(V-o)<0} \mid v_i = r_i \right]$$

of any seller $i$ choosing to exert no influence is zero as

$$\text{sign}(V-o) = \text{sign} \left( s_i o + [1 - s_i]V_i - o \right) = \text{sign} \left( [1 - s_i][V_i - o] \right) = \text{sign}(V_i - o).$$

While BNC is not straightforward for sellers, it is implementable: so long as sellers $j \neq i$ report truthfully, seller $i$ is incentivized to do so.

**Proposition 3.** BNC is budget-balanced and implementable. Moreover, BNC strictly preserves collective property rights.

Relative to SC, BNC trades straightforwardness for budget balance. Thus, but for the practical difficulties discussed above, BNC might be more attractive than SC.

**V.B.2 All-pay Concordance (APC)**

BNC is difficult to implement because it requires the social planner to determine the expected externality payments appropriate for each valuation. These payments can be described by a function $f(v_i - s_i o)$ (possibly idiosyncratic across sellers) with $f(0) = 0$, $f'(x)x > 0$ for all $x$ because sellers with larger announced surplus are pivotal more often (and by larger amounts). Thus the implementability problem of BNC can be seen as arising from the fact that the appropriate functional form of $f$ is unknown. A natural way to address this problem is to assume a simple functional form for $f$ satisfying these properties and hope that this roughly approximates the correct form. One natural candidate is $f(x) = |x|$. 

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All-pay Concordance

1. APC is the Concordance mechanism with taxes equal to the surplus the seller would obtain given her announced reserve from her desired outcome.

2. The sellers’ taxes are redistributed, with each seller $i$ receiving $\frac{s_i}{1-s_j}$ of $j$’s payments.

This suggests the All-pay Concordance (APC) mechanism. In APC each seller pays her full announced surplus and the option creating greater announced net surplus wins. Therefore, this procedure is equivalent to one in which each seller announces a preferred decision and an amount she is willing to pay to obtain this decision and whichever option has greater monetary support is chosen. This latter procedure is a simple adaptation of the standard all-pay auction to the Concordance context. Formally, the APC mechanism is given by $\mathcal{B} = \mathcal{R} = \mathbb{R}_{++}$, $P(o, r) = 1_{o \geq R}$,

$$T_i(o, r) = s_i o P(o, r) - \left| s_i o - r_i \right| + s_i \sum_{j \neq i} \frac{\left| s_j o - r_j \right|}{1-s_j}$$

with Concordance suggested strategies. APC is perfectly budget-balanced, since

$$\sum_i s_i \sum_{j \neq i} \frac{\left| s_j o - r_j \right|}{1-s_j} = \sum_j (1-s_j) \frac{\left| s_j o - r_j \right|}{1-s_j} = \sum_j \left| s_j o - r_j \right|.$$  

This, in addition to the independence of taxes from other sellers’ behavior that APC shares with BNC, seems to indicate that APC may also be more resilient to collusion than SC. In a future draft we hope to investigate this formally by analyzing the susceptibility of APC to mergers and de-mergers. Just as in the BNC mechanism, APC strictly preserves community property rights. Furthermore, APC is a Concordance mechanism by construction, hence the
results of Theorems 1 and 2 apply if buyers and sellers follow suggested strategies.

However, APC is not implementable: suggested strategies for APC cannot form an equilibrium, as this would give sellers negative surplus with near certainty. Thus, it is not clear how relevant Theorems 1 and 2 are to APC.\textsuperscript{46} Further theoretical and experimental research is therefore needed to determine when APC is as efficient and protective of property rights as Concordance mechanisms purport to be.

\textbf{V.B.3 First-price Concordance (FPC)}

For completeness we now show how the final standard auction enforcement mechanism, first-price payments, can be translated to the Concordance setting. The standard first-price auction has bidders declare a value for the object being sold with winners forced to pay their stated values. This gives (well-known) incentives for bidders to understate their valuations. The natural Concordance analog of this approach is to have sellers announce (falsely) a value and then have them pay the associated surplus as a result of obtaining the outcome (sale or no sale) they desire, if this outcome indeed obtains. This mechanism is described in the above box, and given formally by $B = R = \mathbb{R}_{++}$, $P(o, r) = 1_{o \geq R}$, and $T_i(o, r)$ given by

\[
s_i o P(o, r) - \max(0, [s_i o - r_i] 1_{\text{sale}}, [r_i - s_i o] 1_{\text{no sale}}) + s_i \sum_{j \neq i} \max(0, [s_j o - r_j] 1_{\text{sale}}, [r_j - s_j o] 1_{\text{no sale}}) \frac{1 - s_j}{1 - s_j}
\]

with Concordance suggested strategies. Like APC, FPC is not implementable, but is budget-balanced and strictly preserves community property rights.

\textbf{V.B.4 Alternative Enforcement Procedures}

The VCG, Bayes-Nash, all-pay, and first-price enforcement approaches far from exhaust the set of all payment procedures used in auctions. It should be clear how the Concordance

\textsuperscript{46}Some traditional intuitions about the revenue equivalence theorem suggest that the equilibrium outcomes of APC may be related to those of SC, but it is not clear whether these intuitions are valid in this setting.
First-price Concordance

1. FPC is the Concordance mechanism where all positive surplus gained by sellers relative to their disfavored outcome is taxed away.

2. These taxes are then redistributed as with APC

principle can easily be combined with most auction enforcement mechanisms. For example, a number of “approximately incentive compatible” procedures have been proposed in recent years for dealing with complicated package auctions, many using payment schemes that ensure allocations are in, or as near as possible to, the core (Day and Milgrom, 2008; Erdil and Klemperer, Forthcoming). Given the similarities between package auctions with complementarities and the holdout problem, it seems natural to adapt these payment schemes, using the surplus from sale \( s_i o - v_i \) as the basis. However, the core in the holdout problem is nearly always empty.\(^{47}\) Thus, one would have to consider payments that come closest to the core.\(^{48}\) We do not consider such a core-nearest Concordance mechanism here because we believe that in most of the the settings where Concordance mechanisms are likely to be applied, the complexity of the payments involved is likely to outweigh the potential benefits such mechanisms offer.\(^{49}\)

V.C  X-Plurality

A contrast to Concordance mechanisms is a class of mechanism we call \( X\)-plurality. These generalize all those holdout mechanisms we know of that have actually been used in the past.

All \( X\)-plurality mechanisms share with Concordance mechanisms the fact that sellers are suggested to report truthfully and that the offer conditional on sale is divided according to

\(^{47}\)Any coalition excluding a seller who earns a negative surplus when she exerts no influence earns greater surplus than it would were she included.

\(^{48}\)We thank Jeremy Bulow for suggesting this idea to us.

\(^{49}\)However, given that at least some applications (such as spectrum reassembly) involve sophisticated participants, such a mechanism might still prove useful.
**X-Plurality**

1. The buyer is asked to submit the monopsonist-optimal offer against the distribution of minimum offers needed to persuade $X$ percent of the shares to consent and sellers are asked to truthfully report their values.

2. If $\sum_i s_i \mathbb{1}_{o \geq r_i} \geq X$, where $X$ is a pre-specified value, and $o \geq \bar{V}$ where $\bar{V}$ is the lowest possible total community valuation, then the plot is sold. In this case, the buyer pays $o$ and each seller receives $s_i o$. Otherwise no transaction takes place and no money changes hands.

shares; However, no taxes are paid. Moreover, rather than the buyer’s offer being accepted if it exceeds the sum of seller valuations, it is accepted if at least a fraction of shares $X \in [0, 1]$ would “vote for” a sale. That is, sale occurs if there is a collection of sellers constituting at least a fraction $X$ having value-to-share ratios $\frac{v_i}{s_i}$ below the offer $o$.

Standard mechanisms for the holdout problem can be seen as special cases of this rule. As discussed in Section IV, the rules for corporate acquisitions used in virtually all countries and the related rules used for land assembly in various countries at various times have been voting-based, $X$-plurality rules with various thresholds $X$. The standard eminent domain system can be seen as an $X$-plurality system with $X = 0$ as sellers are paid the minimum valuation they could have by revealed preference, the market valuation of their land. The systems of Bagnoli and Lipman (1988), Shavell (2007), and Shapiro and Pincus (2007), which require universal consent, arise as the $X$-plurality mechanism with $X = 1$.

The general $X$-plurality mechanism is formally implemented as $\mathcal{B} = \mathcal{R} = \mathbb{R}_{++}$,

$$P(o, r) = 1 \begin{cases} X \leq \sum_i s_i \mathbb{1}_{o \geq r_i} \end{cases},$$

$$T_i(o, r) = s_i o P(o, r), \quad r^*(v) = v \text{ and } o^* \equiv \arg \max_o (b - o) \tilde{G}^X(o).$$ Here $\tilde{G}^X$ is the cumulative distribution function of the $\ominus_{N,X} \equiv \arg \min_o 1 \begin{cases} X \leq \sum_i s_i \mathbb{1}_{o \geq r_i} \end{cases}$. In the case of equal shares, $\ominus_{N,X}$ is just the $\lceil N(1 - X) \rceil$-th order statistic of the distribution of $\frac{v_i}{s_i}$. The case $X = 1$ encompasses
the mechanisms of Shavell and Shapiro and Pincus; the case $X = \frac{1}{2}$ is equivalent to majority share rule; and the case $X = 0$ corresponds to the typical application of eminent domain.\footnote{Here, we have assumed that the minimal value $V$ is that which is assessed as compensation in a taking. This appears to be reasonable, as in practice takings are often compensated at or below market value—and therefore at the lower bound of possible subjective property valuations (Radin, 1982; Fennell, 2004).}

The strength of $X$-plurality, and the reason it has likely been used so widely, is that it combines the straightforwardness of SC with the budget balance of the other Concordance mechanisms, while being eminently practical and simple. Budget balance is immediate as all revenues generated by the offer are disbursed to the sellers and no taxes on sellers are assessed. Straightforwardness follows from the fact that the mechanism has sellers “voting” in favor of the (generically) unique outcome that yields them weakly positive surplus and voting thus can only increase the probability of this outcome realizing. Implementability follows from the fact that the minimum successful offer that is just sufficient to achieve a sale is that equal to that which is just sufficient to have a fraction $X$ of shares consent. The seller finds it optimal to make the monopsonist’s optimal offer against this “supply curve.”

**Proposition 4.** For all $X$, the $X$-plurality mechanism is budget-balanced, straightforward for sellers and implementable.

However, the $X$-plurality class of mechanisms suffers from two pervasive deficiencies: its inefficiency, potentially both encouraging inefficient and discouraging efficient sales, and its complicated and often unappealing relationship to property rights. We discuss these in turn.

The $X$-plurality mechanism leads to efficient community decision-making given a buyers’ offer when the $X$-th percentile of the share-weighted empirical distribution of value-to-share ratios coincides with the share-weighted average of that distribution.\footnote{Bergstrom (1979a,b) extensively developed in the context of public goods games (effectively) the theory of efficiency (among sellers) of the response to a buyer’s offer in the case of equal (rather than share-based) voting weights and $X = \frac{1}{2}$. In that case in large communities efficiency results if and only if the median of the distribution of value-to-share rations coincides with its mean. Efficiency would require the social planner setting $X$ at the quantile of the distribution corresponding to its mean as suggested by Ledyard and Palfrey (1994, 2002). Of course a number of additional complexities arise in our setting. First, share-weighted voting is not strictly equivalent to equal-weights voting and thus it is the empirical quantile of weighted distribution that is relevant. Second, because communities are not infinitely large it is the empirical rather than population quantiles that are relevant. Third, in small populations where the empirical quantile is...} Thus any efficiency
guarantee for $X$-plurality mechanism would rely on (the social planner) having a clear sense of the distribution of valuations. Anytime such information is not available, the $X$-plurality mechanisms can be highly inefficient—in either direction. If the true, properly weighted distribution is such that the mean is consistently above (below) the $X$-th quantile and the buyer’s value lies between these, most sales (failures to make a sale) will be inefficient except in very small communities of sellers. Thus $X$-plurality seems likely to be inefficient, especially in large communities.

By construction, $X$-plurality preserves the property rights of sellers constituting a fraction $X$ of all shares. It also preserves approximate individual and collective property rights to the extent that the $X$-th empirical quantile is weakly above the mean.\textsuperscript{52} To the extent it is below, collective and approximate individual property rights are violated. In practice this suggests that when $X$ is large, $X$-plurality mechanisms will tend to preserve property rights at least as well as Concordance mechanisms do, if not better, although precise guarantees will depend on assumptions about distributions. However, raising $X$ also inefficiently reduces the number of sales. Thus the class of $X$-plurality mechanisms, in practice, seems to embody an unattractive tradeoff between inefficiency and violation of property rights. Furthermore, there is much scope for vote-buying manipulation, except when $X$ is very small.

V.D Comparison of Holdout Mechanisms

Table 1 presents a comparison among the holdout mechanisms. The comparison also addresses practical factors, such as individuals’ levels of exposure to risk.\textsuperscript{53}

uncertain, because of the Cournot-Myerson-Satterthwaite distortion that exists in small populations it might actually be optimal from an efficiency perspective to set $X$ somewhat below the quantile of the transformed distribution corresponding to mean of the distribution so as to raise the probability of a sale back towards efficiency. Furthermore, for very small communities the empirical distribution may diverge sufficiently from the population distribution that the efficiency properties may depend on different features of the distribution than simply its quantiles. Careful analysis of these issues is beyond the scope of our paper but an interesting direction for future research.

\textsuperscript{52}Except that the share is a share of the actual $V$ not $V_i$.

\textsuperscript{53}In practical implementations of the Concordance mechanisms, it will be necessary for sellers to finance their tax transfers without overflowing their \textit{ex ante} budgets. This is problematic for mechanisms where an individual’s net taxes can be unpredictably negative, as with SC. This issue raises some implementation
<table>
<thead>
<tr>
<th></th>
<th>Finances</th>
<th>Simplicity</th>
<th>Efficiency</th>
<th>Property Rights</th>
<th>Share incentive</th>
<th>Collusion</th>
<th>Practical Issues</th>
</tr>
</thead>
<tbody>
<tr>
<td>SC</td>
<td>Self-financing, asymptotically balanced</td>
<td>Straight-forward for sellers, implementable</td>
<td>Bilateral, asymptotic</td>
<td>Collective, asymp. strict collective, approx. individual</td>
<td>Yes</td>
<td>Moderate?</td>
<td></td>
</tr>
<tr>
<td>BNC</td>
<td>Balanced budget</td>
<td>Implementable</td>
<td>Bilateral, asymptotic</td>
<td>Strict collective, approximate individual</td>
<td>Yes</td>
<td>Low?</td>
<td>Requires detailed knowledge of valuations</td>
</tr>
<tr>
<td>APC</td>
<td>Balanced budget</td>
<td>Complex, possibly unimplementable</td>
<td>Bilateral, asymptotic</td>
<td>Same as BNC</td>
<td>Yes</td>
<td>None?</td>
<td></td>
</tr>
<tr>
<td>FPC</td>
<td>Balanced budget</td>
<td>Very complex, likely unimplementable</td>
<td>Bilateral, asymptotic</td>
<td>Same as BNC</td>
<td>Yes</td>
<td>Very low?</td>
<td></td>
</tr>
<tr>
<td>X-plurality (low X)</td>
<td>Budget balanced</td>
<td>Like SC</td>
<td>Too many sales</td>
<td>None</td>
<td>Yes</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>X-plurality (mid X)</td>
<td>Budget balanced</td>
<td>Like SC</td>
<td>If percentile matches mean</td>
<td>X of shares, approximate individual if efficient</td>
<td>No</td>
<td>High?</td>
<td></td>
</tr>
<tr>
<td>X-plurality (high X)</td>
<td>Budget balanced</td>
<td>Like SC</td>
<td>Holdout: no asymp. gains</td>
<td>Near-perfect individual</td>
<td>Yes</td>
<td>Very high?</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Comparison of holdout mechanisms
The table contains one dimension not discussed above: share incentive compatibility. We omitted this above because for almost all mechanisms misreporting shares does not benefit the buyer in any way because the seller decision is independent of shares.\textsuperscript{54} However, for moderate $X$-level $X$-plurality mechanisms, the seller may gain by allocating all shares to buyers with low valuations. Furthermore, although this is not reported in the table, under eminent domain (0-plurality) the buyer may have an incentive to distort down the “minimum seller valuation” so as to pay compensation below market prices. This has been a common concern in much of the controversy over eminent domain (Castle Coalition, 2009).

Two comparisons are clear. BNC nearly dominates SC, were it practically implementable, which sadly seems unlikely. Moderate $X$-plurality mechanisms dominate those with low $X$, though what exactly constitute “low” and “moderate” is ambiguous.

Narrowing our focus to undominated mechanisms, we have a number of interesting but difficult-to-quantify trade-offs. BNC seems the best of the Concordance mechanisms when it is feasible, while the tradeoffs between the straightforwardness of SC and the other benefits APC and FPC are subtle. APC shows more promise than FPC, but without better theoretical knowledge of their equilibria, as well as the ability of communities to reach equilibrium without infeasible training, SC seems the more attractive alternative at present. However, we suspect that in the long-term some variation on APC may be superior.

Among values for the $X$ in the $X$-plurality mechanism above the best guess of that optimal for efficiency, a simple trade-off between efficiency and property rights appears. Opinions about the appropriate stand on this trade-off are likely to differ widely across honest people. Given that our main goal here is economics, not philosophy, we will not opine on it further except to say that we would guess that to the extent that the current legal environment in the U.S. reveals policy preferences, allowing the holdout problem to persist through a very high $X$ seems unlikely to be a widely accepted policy.

\textsuperscript{54}In fact, the buyer may generally have an interest in reporting shares truthfully as this tends to reduce property rights violations (for Concordance mechanisms), thereby economizing on potential legal costs, or bringing down minimum sale prices (for high $X$-level $X$-plurality mechanisms).
Comparing X-plurality to the Concordance mechanisms is difficult. An X-plurality mechanism is extremely easy to explain and does not require any tax payments, which might conflict with the budget constraints of sellers. Furthermore, the X-fraction-of-shares property rights protection may be attractive to some, though clearly Concordance mechanisms will, under appropriate comparisons of quantiles to means, also satisfy similar properties. However, if one is willing to put faith in complicated institutions, Concordance mechanisms offer a much more attractive set of guarantees about combinations of efficiency, property rights and incentives than any practical X-plurality mechanism could.

To summarize, we believe potential implementers’ preferences likely fall into three camps:

1. Those strongly protective of property rights, who will favor X-plurality with a high X.

2. Those primarily interested in efficiency and simplicity, likely working in contexts where budgets and stakes are very small, who would tend to favor an X-plurality mechanism with an X chosen to roughly maximize efficiency.

3. Those interested in a mix of efficiency and property rights in high-stakes environments where they are willing to expend and require of sellers the resources (material and intellectual) necessary to implement the sophisticated Concordance mechanisms. It is in such contexts that our paper will be most valuable and we suspect such planners will begin with SC and only consider APC, FPC and BNC once more careful research has been done on their effectiveness and practicality.

VI Public Goods and Collaboration

Despite our focus above on holdout, there is an intimate connection between this and the problems of collaboration and public goods. Modifications of our mechanisms above can thus be applied to those problems.
VI.A Binary public goods

Binary public goods, non-rivalrous and non-excludable all-or-nothing public projects, with quasi-linear preferences are essentially equivalent to land assembly, except that the cost of building them is typically assumed to be known publicly while the buyer’s valuation above is the private information of the buyer. It is easy to see that from the perspective of the mechanisms, except in efficiency calculations, this is inessential and In fact, many public projects are privately supplied at privately-known cost.\textsuperscript{55} In all other respects the correspondence is exact. Property rights become voluntary participation, perfect complementarity non-excludability and single-mindedness non-rivalry. True shares are Lindahl prices and actual (approximate) shares are closely connected to the pseudo-Lindahl prices of Bergstrom (1979b), a public authority’s closest approximation to Lindahl based on public information. Thus our mechanism’s guarantee of approximate property rights implies a corresponding binary public goods mechanism that implements generally Bergstrom’s (efficient) pseudo-Lindahl equilibrium. As far as we know, ours is the first mechanism that does so in general.

While strong moral and some economic intuitions point towards approximate property rights as an appealing concept for the holdout problem, the justification of approximate non-coercion for public goods games is less obvious. However, a long tradition dating back to Lindahl has emphasized the fairness of this approach and thus we consider our mechanism something of a contribution to the theory of public goods. We suspect the reason we were able to make this contribution is that we took as our starting point the less general problem of binary, quasi-linear public goods games and exploited the special structure they imply. As we will see some of the insights gained from this context can be directly generalized, but that the connection to the fully general setting is not immediately apparent.

\textsuperscript{55}A perfect example is the reverse of land assembly: land reform. All of our mechanisms apply just as easily there: the seller makes a take-it-or-leave-it offer to a community of tenant farmers who are coerced to participate in a Concordance mechanism for purchasing the land.
Straightforward Collaboration

1. Collaborators agree on a demand function $Q$ and are assigned shares $\{s_i\}_{i=1}^{N}$.

2. Each seller $i$ submits a cost $c_i$.

3. $P^* (C)$ is charged to consumers and sellers receive their share of revenue.

4. Each seller pays the Pigouvian tax

   \[
   (1 - s_i) \left( [P^* (C_i) - C_i] Q [P^* (C)] - [P^* (C) - C_i] Q [P^* (C)] \right).
   \]

5. Each receives a refund analogous to that in SC above.

VI.B Collaboration

Collaboration inherently involves sophisticated sellers and diffuse consumers, so it calls either for a regulator or a mechanisms that forces buyers to be price taker. This requires retooling the concordance mechanisms to put sellers in charge of bargaining. Thus we need the market demand function $Q$ to be common knowledge among sellers and the social planner.\textsuperscript{56} However, given this, our mechanism requires only minor retooling, as laid out for the straightforward implementation in the box above, to be applied to collaboration.

In Appendix E we describe this mechanism, as well as its properties and alternatives to it, in greater detail. All of the results and description are almost perfectly analogous, in the spiring of Bulow and Roberts (1989), to the holdout problem.

\textsuperscript{56}While this is a common assumption in industrial organization, it poses a substantial problem for the practical implementation of the mechanism. The mechanism explicitly uses this commonly known demand function to construct the payoffs of each agent; thus if the demand is not explicitly commonly known it would be hard to implement the mechanism. Of course this practical difficulty could be overcome by randomizing prices, or using structural econometrics (Baker and Bresnahan, 1988; Berry et al., 1995), to measure industry demand to all parties’ satisfaction before implementing the mechanism.
VI.C General public goods

Applying collaboration mechanisms to continuous public goods problems is simple if taxpayer preferences are quasi-linear and known up to a term affine shift $v_i f(E)$ where $v_i$ is the idiosyncratic valuation, $f$ is a smooth concave function and $E$ is the total expenditure on the public good. Of course this is still a very specific context for public goods as it disallows income effects and heterogeneity in the curvature of valuation of the good.

A more general direction for extending the concordance principle to quasi-linear but general valuation functions would be to define no influence as submitting $v(e) = s_i e$. Influence potentially subject to taxation would then be defined relative to that benchmark. This is an interesting direction for future research and would allow the extension of collaboration mechanisms to, for example, general cost functions.

VII Conclusion

This paper makes two contributions. First, we bring the holdout problem to the attention of the market design community, both by emphasizing its importance and the lack of a satisfying solution thus far, and by bringing together disparate strands of literature and systematizing our knowledge of the holdout problem. Second, we introduce a potential framework for solving the holdout problem, which maintains a semblance of property rights while achieving substantial efficiency. To our knowledge, our paper is the first to propose a novel market design solution to holdout. Hence, we expect—and in fact, hope—that our work will provide neither the most definitive comparison among the approaches we discuss, nor the final answer to the problem. There are many interesting directions for future research.

Our analysis could naturally be extended in a number of ways. More analytical, computational and experimental work is needed to understand the behavior of the APC and FPC mechanisms, as well as the incentives for and impact of collusion in various mechanisms dis-
cussed above. More thought should be given to precisely implementing BNC.\textsuperscript{57} It would be useful to understand better the efficiency-optimal choice of $X$ for $X$-plurality mechanisms, particularly how this varies with community size and distributions. Fully understanding the reasons, philosophical, legal and economic, why property rights protections are desirable properties of a holdout mechanism would help clarify what compromises on these rights are reasonable. Extension of the mechanisms and related property rights guarantees to broader public goods games is an interesting theoretical design problem. Measuring more carefully than our embarrassingly casual back-of-the-envelope guess the real-world losses of social value due to holdout problems is an important empirical question.

Many relatively minor extensions of our mechanism could expand their range of applicability. Concordance mechanisms place full property rights into community hands, but it would be simple, and natural in many eminent domain contexts, to place property rights partially into the buyer’s hands; it is known (Segal and Whinston, Forthcoming) that this helps mitigate the residual Cournot-Myerson-Satterthwaite distortion. Our Concordance mechanisms all require sellers to make tax payments, which are partially refunded, for enforcing on the community their preferences about sales. In the real world, sellers often face budget constraints that may make this feature unattractive.\textsuperscript{58}

Finally, one broader and more-ambitious extension suggests itself: we restricted our attention to a case of perfect complements and assumed no competition between aggregate land plots. In many practical settings a contiguous block of land must be assembled, but several such blocks may compete to host such a project; some of the competing collections may even overlap. This problem of collaboration nested within in competition raises a number of interesting questions. Both in Cournot’s collaboration-competition model and in mechanism design, how fast must competition grow relative to collaboration for efficiency to improve

\textsuperscript{57}Fine-tuning the Concordance mechanisms and their explanation to sellers will require experimental research. A field implementation of the system will be a crucial test of concept.

\textsuperscript{58}Designs (similar to those of Pal and Vohra (2009) for auctions) that come close to preserving the attractive properties of Concordance mechanisms while accommodating bidders with privately known budget constraints would be a challenging but practically important extension of our work.
(or worsen) with size? What are natural mechanisms for a setting of holdout combined with competitive procurement? More generally: how does the Concordance principle extend to imperfect complements?

References


Appendix

A The Concordance Principle

In this section, we prove the two parts of Theorem 1.

1 Bilateral Efficiency

The outcome of any Concordance mechanism $M \in \mathcal{C}$ is exactly identical to that of its bilateral form $M'$ with the distribution of the seller’s value being that of $V$. This follows directly from the fact that the truthfulness is suggested for all sellers in $M$ and for the single seller in $M'$, while in both $M$ and $M'$ the buyer’s suggested offer is the monopsonist optimal price facing the distribution of $V$, proving the first half of the theorem.
2 Asymptotic Efficiency

We let \( \{x^n_i\} \) represent the i.i.d. process in the theorem statement, and write \( \mu \) and \( \sigma^2 \) for the (strictly positive) mean and (strictly positive) variance of this process, respectively. As

\[
V^n = \sum_{i=1}^{n} v^n_i = \sum_{i=1}^{n} x^n_is^n_i
\]

and \( \sum_{i=1}^{n} s^n_i = 1 \), we have

\[
E[V^n] = \mu, \quad \text{Var}[V^n] < \frac{M^2\sigma^2}{n}
\]

by the share bound and i.i.d. hypotheses.

To show the result, it suffices to demonstrate that the total inefficiency of \( M^n \) vanishes as \( n \to \infty \).

The buyer is instructed to make the monopsonist’s optimal offer against supply function \( \overline{G}_n(o) \equiv \text{Prob}[o \geq V^n] \), or equivalently against inverse supply \( S_n(q) \equiv \overline{G}_n^{-1}(q) \). That is, the buyer maximizes

\[
q (b - S_n(q))
\]

over \( q \). We let \( \tilde{q}_n(b) \) be the optimal choice of \( q \) in (5), for a buyer with value \( b \).

We first examine the case in which \( b > \mu \). We show that as \( n \) becomes large, a buyer with value \( b \) chooses a probability of sale approaching 1.

Lemma 1. For any fixed \( b > \mu \), we have \( \tilde{q}_n(b) \to 1 \) as \( n \to \infty \).

Proof. By the one-sided Chebyshev inequality, we have for any \( \alpha > 0 \)

\[
\text{Prob}[V^n - \mu \geq \alpha] \leq \frac{M^2\sigma^2}{M^2\sigma^2 + n\alpha^2},
\]

which vanishes as \( n \to \infty \). It follows immediately that \( S_n(q) \to \mu \) as \( n \to \infty \) (pointwise) for any \( q \). Thus, for any fixed \( \epsilon > 0 \), we have

\[
(q + \epsilon) (b - S_n(q + \epsilon)) > q(b - S_n(q))
\]

for \( n \) sufficiently large (since \( (S_n(q + \epsilon) - S_n(q)) \to 0 \)), hence we have \( \tilde{q}_n(b) \to 1 \) as \( n \to \infty \), for any given \( b \).

Note that because \( b \geq o \) and a sale takes place only when \( o \geq V \), inefficient sales will never take place. Thus, the total inefficiency of \( M^n \) in cases when \( b > \mu \) is bounded above

\(^{59}\)This is sufficient because the gains from trade are bounded away from zero.

\(^{60}\)The case in which \( b < \mu \) is simpler, as we discuss below.
by the potential surplus created by failed efficient sales:

$$\int_\mu^\infty (1 - \tilde{q}_n(b))(b - \mu)h(b) \, db,$$

(6)

where $h(b)$ is the probability density function of $b$. In order for the efficiency of $\mathcal{M}^n$ to be well-defined (which we assume), $b$ must have a finite upper-tail integral. We therefore have

$$\int_\mu^\infty (1 - \tilde{q}_n(b))(b - \mu)h(b) \, db \leq \int_\mu^\infty (b - \mu)h(b) \, db < \infty$$

for any $\bar{b} > \mu$. Thus, to bound (6), it suffices to show that

$$\int_\mu^{\bar{b}} (1 - \tilde{q}_n(b))(b - \mu)h(b) \, db \leq (\bar{b} - \mu) \int_\mu^{\bar{b}} (1 - \tilde{q}_n(b))h(b) \, db$$

(7)

vanishes as $n \to \infty$, for any fixed $\bar{b} > \mu$. We now show this fact.

**Lemma 2.** The integral

$$\int_\mu^{\bar{b}} (1 - \tilde{q}_n(b))h(b) \, db$$

(8)

vanishes as $n \to \infty$.

**Proof.** As the integrand of (7) lies in $[0, 1]$ for all $n$ and $b$, we may take the limit of (8) as $n \to \infty$ and apply the dominated convergence theorem. The claim then follows since $\tilde{q}_n(b) \to 1$ (pointwise) as $n \to \infty$ by Lemma 1.

Lemma 2 shows that the right side of (7) vanishes as $n \to \infty$. We therefore see that, as $n \to \infty$, $\mathcal{M}^n$ is fully efficient for buyers with values $b > \mu$. As the buyer never offers more than $b$, it is quick to show that inefficiency of $\mathcal{M}^n$ vanishes (as $n \to \infty$) when $b < \mu$.62 Thus, we have the desired result.

**B Groves Holdout Mechanisms**

We now derive our main results for the SC mechanism, as special cases of more general results for the full class of Groves mechanisms for the holdout problem.

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61If (7) vanishes for any $\bar{b} > \mu$, then we must have

$$\int_\mu^\infty (1 - \tilde{q}_n(b))(b - \mu)h(b) \, db = \int_\mu^{\bar{b}} (1 - \tilde{q}_n(b))(b - \mu)h(b) \, db + \int_\mu^\infty (1 - \tilde{q}_n(b))(b - \mu)h(b) \, db \to 0$$

as $\bar{b} \to \infty$, whence we see that (6) vanishes as $n \to \infty$.

62As trade is always inefficient when $b < V$, the inefficiency of $\mathcal{M}^n$ in the case $b < \mu$ is bounded by an integral with finite support $[V, \mu]$. An argument directly analogous to that for the case $b > \mu$ shows that the integrand (i.e. the inefficiency for fixed $b < \mu$) vanishes pointwise, hence the inefficiency of $\mathcal{M}^n$ vanishes globally by the dominated convergence theorem.
Definition 11. A mechanism $\mathcal{M} = \{B, R, P, T\}$ with $B = R = \mathbb{R}_{++}$ is a Groves Holdout (GH) mechanism if:

- $P(o, r) = 1_{o \geq R}$;
- $T_\beta(o, r) = -oP(o, r)$;
- $T_i(o, r) = s_i o P(o, r) + \sum_{j \neq i} (s_j o P(o, r) + r_j (1 - P(o, r))) + h_i(o, r_{-i})$, where, $h_i(o, r_{-i})$ is a function independent of the reserve reported by seller $i$.

The first condition ensures that transfer occurs if and only if the buyer’s offer exceeds the collective reserve, while the second simply requires that the buyer pay her offer in the case of transfer. The third condition holds most of the content of the definition; it means that for all sellers $i$, the Groves tax rule $\tau_i(o, r) \equiv T_i(o, r) - s_i o P(o, r)$ takes the form of the transfer rule of a classical Groves (1973) mechanism (citation year suppressed in the sequel).\textsuperscript{63} Since a GH mechanism is completely specified by either $h_i(o, r_{-i})$ or $\tau_i(o, r)$, we use these two functions interchangeably according to convenience.

The suggested strategies for a GH mechanism are $r^*(v) \equiv v$ and $o^*(b) = \arg\max_{o} (b - o) G(o)$.\textsuperscript{64}

1 Basic Properties

It follows immediately from standard results on Groves mechanisms that $r^*(v)$ is optimal, i.e. any GH mechanism is straightforward for sellers.\textsuperscript{65} Indeed, it is quick to compute that in such a mechanism the ex post utility of seller $i$ is given by

$$T_i(o, r) + v_i(1 - P(o, r)) = o P(o, r) + \left(v_i + \sum_{j \neq i} r_j\right)(1 - P(o, r)) + h_i(o, r_{-i}).$$

It follows immediately that the seller $i$ wants sale to happen if and only if $o \geq v_i + \sum_{j \neq i} r_j$, which is exactly the outcome implemented if seller $i$ reports $r_i = v_i$.

Moreover, the suggested strategy of a GH mechanism is optimal for the buyer. If the sellers follow the strategy $r^*$, sellers will accept the buyer’s offer whenever $o \geq V$. Thus the buyer’s expected payoff in this case is $(b - o) G(o)$, hence $o^*$ is her best offer by construction.

The preceding observations are summarized in the following proposition.

Proposition 5. Any GH mechanism is straightforward for sellers and implementable.

\textsuperscript{63}Note that when refunds are present, this “tax” rule incorporates both the tax and refund components of the mechanism, as we detail further below.

\textsuperscript{64}Here, as in the main text, $G$ is the cumulative distribution function of $V$; that is $o^*(b)$ is the monopsonist optimal price facing an supply function $G(o)$.

\textsuperscript{65}Note that our normalization of seller utility differs slightly from that which would be expected in a standard Groves framework. We take the utility in case of no sale to be $v_i$. It would be more in line with the mechanism design literature—but substantially less intuitive—to treat the case of no sale as a baseline giving utility 0. Then, the utility in case of sale would be given by $s_i o - v_i$. To convert to this alternative normalization, it suffices to (additively) renormalize each function $h_i(o, r_{-i})$ by $(1 - s_i) R_i$. 

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2 Uniqueness of the Groves Holdout Class

We now prove that the Groves Holdout class is unique: any mechanism for the holdout problem, which is straightforward for sellers and bilaterally efficient is isomorphic to a GH mechanism. We begin with a lemma, which extends Theorem 3 of Green and Laffont (1977) to our setting.\(^66\)

**Lemma 3.** _Any (direct revelation) mechanism for the holdout problem, which_

- _induces a Pareto-optimal outcome amongst sellers (conditional upon share division and the buyer’s offer \(o\)) and_
- _has truthful reporting as a dominant strategy for each seller_

_is a Groves Holdout mechanism._\(^67\)

**Proof.** It suffices to show that any mechanism satisfying the conditions of the lemma has the property that

\[
\tau_i(o, r_i, r_{-i}) - \tau_i(o, \hat{r}_i, r_{-i}) = (P(o, r_i, r_{-i}) - P(o, \hat{r}_i, r_{-i})) \sum_{j \neq i} (s_j o - r_j). \tag{9}
\]

If this property fails, then there exist \(r_i, \hat{r}_i,\) and \(r_{-i}\) such that

\[
\tau_i(o, r_i, r_{-i}) - \tau_i(o, \hat{r}_i, r_{-i}) = (P(o, r_i, r_{-i}) - P(o, \hat{r}_i, r_{-i})) \sum_{j \neq i} (s_j o - r_j) + \epsilon
\]

for some \(\epsilon \neq 0\). Without loss of generality, we assume that \(\epsilon > 0\). If \(P(o, r_i, r_{-i}) = P(o, \hat{r}_i, r_{-i})\), then clearly truthful play is not a dominant strategy for seller \(i\) anticipating other sellers reporting truthfully, as \(i\) should announce reservation value \(v_i\) when \(v_i = \hat{r}_i\). Thus, we may assume that \(P(o, r_i, r_{-i}) \neq P(o, \hat{r}_i, r_{-i})\). By Pareto optimality, we have \(P(o, r_i, r_{-i}), P(o, \hat{r}_i, r_{-i}) \in \{0, 1\}\). We assume that \(P(o, r_i, r_{-i}) = 1\); the case \(P(o, r_i, r_{-i}) = 0\) follows analogously.

Let \(\hat{r}_i' = s_i o - \sum_{j \neq i} (s_j o P(o, \hat{r}_i, r_{-i}) + r_j (1 - P(o, \hat{r}_i, r_{-i}))) - \delta\) for some small \(\delta > 0\).\(^68\) Then by Pareto optimality \(P(o, \hat{r}_i', r_{-i}) = 0 = P(o, \hat{r}_i, r_{-i})\); it follows that \(\tau_i(o, \hat{r}_i', r_{-i}) =

\(^66\) Even with the renormalization described in Footnote 65, Theorem 3 of Green and Laffont (1977) does not directly apply to our setting. The approach of Green and Laffont (1977) requires at least three outcomes, and in our setting there are only two: sale and no sale. Nonetheless, the basic approach of Green and Laffont (1977) generalizes directly to the argument we give in the proof of Lemma 3.

\(^67\) By a Pareto-optimal outcome (conditional upon share division and the buyer’s offer \(o\)), we mean a choice of \(P(o, r)\) such that \(\sum_{i=1}^{N} (s_i o P(o, r) + r_j (1 - P(o, r)))\) is maximized.

\(^68\) This \(\delta\) is only needed to prevent indifference in the sale decision.
\[ \tau_i(o, \hat{r}_i, r_{-i}).69 \text{ But this implies that} \\
\tau_i(o, r_i, r_{-i}) - \tau_i(o, \hat{r}_i, r_{-i}) = \tau_i(o, r_i, r_{-i}) - \tau_i(o, \hat{r}_i, r_{-i}) \\
= (P(o, r_i, r_{-i}) - P(o, \hat{r}_i, r_{-i})) \sum_{j \neq i} (s_j o - r_j) + \epsilon \\
= \left( \sum_{j \neq i} (s_j o P(o, r_i, r_{-i}) + r_j (1 - P(o, r_i, r_{-i}))) \right) - \left( \sum_{j \neq i} (s_j o P(o, \hat{r}_i, r_{-i}) + r_j (1 - P(o, \hat{r}_i, r_{-i}))) \right) + \epsilon \\
= (1 - s_i) o + (\hat{r}_i - s_i o + \delta) + \epsilon \\
\geq \hat{r}_i - s_i o + \epsilon. \\
\]

But then, we have
\[ \tau_i(o, r_i, r_{-i}) + s_i o > \tau_i(o, \hat{r}_i, r_{-i}) + \hat{r}_i', \]
from which it follows that truthful reporting is not a dominant strategy for \( i \) when \( i \) anticipates truthful reports from other sellers and \( v_i = \hat{r}_i'.70 \)

Now, we can prove the uniqueness of the Groves Holdout class.

**Proposition 6.** Let \( \mathcal{M} \) be a (direct revelation) mechanism for the holdout problem, which

1. is straightforward for sellers,
2. induces a Pareto-optimal outcome amongst sellers (conditional upon share division and the buyer’s offer \( o \)), and
3. is bilaterally efficient relative to a take-it-or-leave-it offer.71

Then, \( \mathcal{M} \) is isomorphic to a GH mechanism \( \mathcal{M}' \).

**Proof.** We denote by \( r^*(v) \) the strategy suggested in mechanism \( \mathcal{M} = \{\mathbb{R}_{++}, \mathbb{R}_{++}, P, T\} \) for a seller with value \( v \). Now, we take \( \mathcal{M}' \) to be the mechanism
\[ \mathcal{M}' = \{\mathbb{R}_{++}, \mathbb{R}_{++}, P(o, r^*(r)), T(o, r^*(r))\}. \]

It follows quickly from the straightforwardness of \( \mathcal{M} \) that \( \mathcal{M}' \) has truthful reporting as a dominant strategy for each seller.72 Additionally, since \( \mathcal{M} \) induces a Pareto-optimal outcome amongst sellers under the suggested strategies, \( \mathcal{M}' \) induces a Pareto-optimal outcome amongst sellers under truthful reporting. Then, it follows from Lemma 3 that \( \mathcal{M}' \) is a GH mechanism.

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69If \( \tau_i(o, \hat{r}_i', r_{-i}) \neq \tau_i(o, \hat{r}_i, r_{-i}) \), then we have found \( \hat{r}_i', \hat{r}_i, \) and \( r_{-i} \), such that \( P(o, \hat{r}_i', r_{-i}) = P(o, \hat{r}_i, r_{-i}) \) and either \( \tau_i(o, \hat{r}_i', r_{-i}) > \tau_i(o, \hat{r}_i, r_{-i}) \) or \( \tau_i(o, \hat{r}_i', r_{-i}) < \tau_i(o, \hat{r}_i, r_{-i}) \). But in either of these cases, truthful reporting is not always dominant. Indeed, in the first case, \( i \) should report \( \hat{r}_i' \) when \( v_i = \hat{r}_i' \); in the second case \( i \) should report \( \hat{r}_i \) when \( v_i = \hat{r}_i \).

70When \( v_i = \hat{r}_i' \), seller \( i \) should announce the reservation value \( r_i \) instead.

71By a Pareto-optimal outcome (conditional upon share division and the buyer’s offer \( o \)), we mean a choice of \( P(o, r) \) such that \( \sum_{i=1}^{N} (s_i P(o, r) + r_i (1 - P(o, r))) \) is maximized.

72The proof of this statement exactly follows the first part of the proof of Theorem 5 of Green and Laffont (1977), hence we omit it.
Finally, we observe that $r^*$ must be univalent on the domain of values $[v, v']$. To see this, we suppose the contrary, i.e. that there are two valuation profiles $v \neq v'$ such that $r^*(v) = r^*(v')$. Now, we fix the buyer’s value at $b$, and observe that the (optimal) buyer offer $o$ in $\mathcal{M}$ submitted when seller values are given by $v$ must be the same as that submitted when seller values are $v'$. But this contradicts the bilateral efficiency of $\mathcal{M}$, since $P(o, r^*(v)) = P(o, r^*(v'))$ but $v \neq v'$. Thus, $r^*$ must be univalent.

From the univalence of $r^*$, we construct the inverse $(r^*)^{-1}$. Composing this function with the reserves $r$ submitted to $\mathcal{M}$ returns the seller values $v = (r^*)^{-1}(r)$. As $\mathcal{M}$ is equivalent to $\mathcal{M}'$ run on $v = (r^*)^{-1}(r)$, we have demonstrated the desired isomorphism.

3 Optimal Tax Refunds

Closely following the approach of Cavallo (2006), we now present a method for redistribution of the taxes collected by a Groves Holdout mechanism $\mathcal{M}$ with a (weakly) negative tax burden on sellers $\tau_i(o, r) \leq 0$, and show that this redistribution method is optimal in a certain natural sense. We let

$$\sigma_i(o, r_{-i}) \equiv \min_{\hat{r}_i} \sum_{j=1}^{N} \tau_j(o, \hat{r}_i, r_{-i})$$

be the lower bound on tax surplus computed across all possible reports $\hat{r}_i$ of seller $i$. We then set

$$\tau_i'(o, r) \equiv \tau_i(o, r) - s_i \sigma_i(o, r_{-i}).$$

**Proposition 7.** The GH mechanism $\mathcal{M}'$ with Groves tax equal to $\tau_i'(o, r)$ is self-financing.

**Proof.** This is immediate, as we have

$$\sum_{i=1}^{N} \tau_i'(o, r) = \sum_{i=1}^{N} (\tau_i(o, r) - s_i \sigma_i(o, r_{-i}))$$

$$= \sum_{i=1}^{N} \left( \tau_i(o, r) - s_i \min_{\hat{r}_i} \sum_{j=1}^{N} \tau_j(o, \hat{r}_i, r_{-i}) \right)$$

$$= \sum_{i=1}^{N} \left( \sum_{j=1}^{N} s_j \right) \tau_i(o, r) - \sum_{i=1}^{N} s_i \min_{\hat{r}_i} \sum_{j=1}^{N} \tau_j(o, \hat{r}_i, r_{-i})$$

$$= \sum_{i=1}^{N} s_i \sum_{j=1}^{N} \tau_j(o, r) - \sum_{i=1}^{N} s_i \min_{\hat{r}_i} \sum_{j=1}^{N} \tau_j(o, \hat{r}_i, r_{-i})$$

$$= \sum_{i=1}^{N} s_i \sum_{j=1}^{N} \left( \tau_j(o, r) - \min_{\hat{r}_i} \sum_{j=1}^{N} \tau_j(o, \hat{r}_i, r_{-i}) \right) < 0.$$
ant to domain information that does not apply identically to every seller. Since the space of seller values will always be compact and simply connected, a direct adaptation of the proof of Theorem 6 of Cavallo (2006) shows the following proposition.

**Proposition 8.** The GH mechanism \( \mathcal{M} \) runs a minimal budget imbalance among all ex post individually rational, self-financing, bilaterally efficient, \( \mathcal{M} \)-surplus-anonymous mechanisms.

### 4 Constructing Straightforward Concordance

We denote by \( P_{-i}(o,r_{-i}) \equiv 1_{o \geq R_i} \) the sale decision, which the community would make absent \( i \), and consider the Groves Holdout mechanism \( \mathcal{M}^{\text{VCG}} \) with \( h_{i}(o,r_{-i}) \) given by

\[
    h_{i}^{\text{VCG}}(o,r_{-i}) \equiv -(1 - s_i) (R_i + (o - R_i)P_{-i}(o,r_{-i}))
\]

The function \( h_{i}^{\text{VCG}}(o,r_{-i}) \) has associated tax function

\[
    \tau_{i}^{\text{VCG}}(o,r) \equiv \sum_{j \neq i} (s_j oP(o,r) + r_j (1 - P(o,r))) + h_{i}^{\text{VCG}}(o,r_{-i})
\]

\[
= \sum_{j \neq i} (s_j oP(o,r) + r_j (1 - P(o,r))) - (1 - s_i) (R_i + (o - R_i)P_{-i}(o,r_{-i}))
\]

\[
= \sum_{j \neq i} ((s_j o - r_j)P(o,r)) - (1 - s_i) (o - R_i)P_{-i}(o,r_{-i})
\]

\[
= (1 - s_i) (o - R_i)P(o,r) - (1 - s_i) (o - R_i)P_{-i}(o,r_{-i})
\]

\[
= \begin{cases} 
    -(1 - s_i) |o - R_i| & P(o,r) = 1 - P_{-i}(o,r_{-i}) \\
    0 & P(o,r) = P_{-i}(o,r_{-i})
\end{cases}
\]

We therefore see that \( \mathcal{M}^{\text{VCG}} \) implements the tax rule of the SC mechanism \( \mathcal{M}^{\star} \) (without a refund).

### 4.a Uniqueness of the VCG Tax

We now prove that the SC tax is the unique enforcement mechanism which can be used in a straightforward Concordance mechanism. Specifically, we use Proposition 6 to show that every straightforward Concordance mechanism is a GH mechanism with \( h_{+} \), \( h_{i}^{+} \), and \( h_{i}^{-} \) everywhere nonnegative. This pins down the SC enforcement uniquely, although the refund mechanism may still be modified while preserving straightforwardness and Concordance.

**Proof of Proposition 2.** We suppose that \( \mathcal{M} \) is a straightforward Concordance mechanism with suggested strategy \( r^{\star}(v) = v \). Then, \( \mathcal{M} \) is bilaterally efficient relative to a take-it-or-leave-it offer (by Theorem 1), and induces a Pareto-optimal outcome among sellers (since the decision rule of \( \mathcal{M} \) is given by \( P(o,r) = 1_{o \geq R} \)). Then, by Proposition 6, there is a GH mechanism \( \mathcal{M}' \) isomorphic to \( \mathcal{M} \). Moreover, by the proof of Proposition 6, that isomorphism is given by the map \( (r^{\star})^{-1} = 1 \), hence \( \mathcal{M} = \mathcal{M}' \) is a GH mechanism.
We decompose the function \( h_t^+(o, r_{-i}) \) in the transfer function \( T_t(o, r) \) of \( \mathcal{M} \) in the form

\[
h_t^+(o, r_{-i}) = h_i^{\text{VCG}}(o, r_{-i}) + \hat{h}_i(o, r_{-i});
\]

it suffices to show that \( \hat{h}_i(o, r_{-i}) \geq 0 \) for all \((o, r_{-i}) \in \mathcal{O} \times \mathcal{R}^{N-1}\). But this is immediate since by the Concordance principle we must have

\[
s_i o P(o, r) + \hat{h}_i(o, r_{-i}) \geq s_i o P(o, r) + \tau_i^{\text{VCG}}(o, r) + \hat{h}_i(o, r_{-i}) = T_t(o, r) \geq s_i o P(o, r)
\]

when valuation \( s_i o \) is submitted by seller \( i \).

Additionally, we can now prove a result, which combines with Proposition 6 to prove the remark stated in Footnote 40.

**Proposition 9.** Suppose that \( \mathcal{M} \) is a GH mechanism, which preserves approximate individual property rights. Then, the function \( h_i(o, r_{-i}) \) in the transfer function \( T_t(o, r) \) of \( \mathcal{M} \) can be decomposed in the form

\[
h_i(o, r_{-i}) = h_i^{\text{VCG}}(o, r_{-i}) + \hat{h}_i(o, r_{-i})
\]

with \( \hat{h}_i(o, r_{-i}) \) bounded below by

\[
\hat{h}_i(o, r_{-i}) \geq \begin{cases} -s_i (o - R_i) & \text{if } P_{-i}(o, r_{-i}) = 1 \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** We fix a seller \( i \) and suppose that the buyer’s value is \( b \) and that the values of the sellers \( j \neq i \) are given by \( v_{-i} \). We set \( o = o^*(b) \) and \( r_{-i} = r^*(v_{-i}) \).

Since \( \mathcal{M} \) preserves approximate individual property rights, there must be some \( r_i \in \mathcal{R} \) such that

\[
s_i o P(o, r_i, r_{-i}) + \tau_i^{\text{VCG}}(o, r) + \hat{h}_i(o, r_{-i}) = T_t(o^*(b), r_i, r^*(v_{-i})) \geq \frac{s_i \sum_{j \neq i} v_{j}}{1 - s_i} \text{P}(o^*(b), r_i, r^*(v_{-i}))
\]

\[
= \frac{s_i \sum_{j \neq i} r_{j}}{1 - s_i} \text{P}(o, r_i, r_{-i}) = s_i R_i \text{P}(o, r_i, r_{-i}).
\]

Reorganizing this expression gives

\[
s_i (o - R_i) P(o, r_i, r_{-i}) \geq - (\tau_i^{\text{VCG}}(o, r) + \hat{h}_i(o, r_{-i}))
\]

\[
= \begin{cases} (1 - s_i) |o - R_i| - \hat{h}_i(o, r_{-i}) & \text{if } P_{-i}(o, r_{-i}) = 1 - P_{-i}(o, r_{-i}) \\ -\hat{h}_i(o, r_{-i}) & \text{if } P_{-i}(o, r_{-i}) = P_{-i}(o, r_{-i}). \end{cases}
\]

(10)

If \( P_{-i}(o, r_{-i}) = 0 \), then (10) cannot hold if \( \hat{h}_i(o, r_{-i}) < 0 \). If instead \( P_{-i}(o, r_{-i}) = 1 \), then we obtain exactly the desired bound on \( \hat{h}_i(o, r_{-i}) \).

\(^{74}\)In the first case, the left side of (10) vanishes, and in the second case the left side of (10) is weakly negative. Meanwhile, if \( \hat{h}_i(o, r_{-i}) < 0 \), then the right side of (10) is strictly positive.
4.b The SC Refund

Next, we obtain the full form of SC by implementing the optimal redistribution rule obtained in Section 3.

Computing the lower bound on $\tau_{i}^{VCG}(o, r)$ gives

$$\sigma_{i}^{VCG}(o, r-i) \equiv \min_{\hat{r}_i} \sum_{j=1}^{N} \tau_{j}^{VCG}(o, \hat{r}_i, r-i) = -\min_{\hat{r}_i} \sum_{j=1}^{N} \left(1_{(\hat{R}_j - o)(\hat{R}_j - o)}(1 - s_j)|o - \hat{R}_j|\right),$$

where $\hat{R}$ and $\hat{R}_j$ are computed as $R$ and $R_j$ with $\hat{r}_i$ substituted in place of $r_i$. Applying the optimal refund of $\tau_{i}^{VCG}(o, r)$, we then obtain exactly the Groves tax used in the SC mechanism,

$$\tau_{i}^{*}(o, r) \equiv \tau_{i}(o, r) - s_i \sigma_{i}^{VCG}(o, r-i).$$

Following this construction, we see that Proposition 1 follows from Propositions 5 and 7. Since SC is also a Concordance mechanism, Theorem 1 applies in dominant-strategy equilibrium.

C Alternative Mechanisms

1 Bayes-Nash Concordance (BNC)

Proof of Proposition 3. Implementability is shown in Subsubsection V.B.1. Budget balance follows because

$$\sum_{j} (1 - s_j) E_{v_{-j}} \left[|V_j - o|1_{(V_j - o)(V_j - o)<0} | v_j = r_j\right]$$

$$= \sum_{j} \left(\sum_{i \neq j} s_i \right) E_{v_{-j}} \left[|V_j - o|1_{(V_j - o)(V_j - o)<0} | v_j = r_j\right]$$

$$= \sum_{i} s_i \sum_{j \neq i} E_{v_{-j}} \left[|V_j - o|1_{(V_j - o)(V_j - o)<0} | v_j = r_j\right].$$

The fact that BNC preserves collective property rights is immediate because

- each seller $i$ announces $r_i = v_i$ in (Bayes-Nash) equilibrium,
- transfer occurs in BNC if and only if $o \geq R = \sum_i r_i = \sum_i v_i = V$, and
- all tax revenues collected in BNC are shared amongst the sellers.

The approximate property rights and efficiency claims follow from Theorems 1 and 2 since truthful revelation is (Bayes-Nash) equilibrium behavior for sellers.
2 First-price Concordance (FPC)

To see that FPC is budget-balanced, it suffices to observe that

\[ \sum_j \max (0, [s_j o - r_j] 1_{sale}, [r_j - s_j o] 1_{no sale}) \]
\[ = \sum_j (1 - s_j) \max (0, [s_j o - r_j] 1_{sale}, [r_j - s_j o] 1_{no sale}) \]
\[ = \sum_j \left( \sum_{i \neq j} s_i \right) \max (0, [s_j o - r_j] 1_{sale}, [r_j - s_j o] 1_{no sale}) \]
\[ = \sum_i s_i \sum_{j \neq i} \max (0, [s_j o - r_j] 1_{sale}, [r_j - s_j o] 1_{no sale}) \]

D Public Goods and Collaboration

This appendix first establishes certain relationships and conditional equivalences among holdout, collaboration and public goods problems. It then briefly analyzes the Straightforward Collaboration mechanism outlined in the text. Finally, it describes in just a bit more detail the other mechanisms for the collaboration problem mentioned in the text.

1 Translation

1.a Quasi-linear, binary public goods

\( N \) members of a community have values \( v_i \) for a public good and their utility is quasi-linear in a private numeraire good (money). The good is non-excludable and non-rival in the sense that if it is created all members of the community gain their value and none gain any value if it is not (fully) created; this is analogous to perfect complements. The cost of creating the public good \( C \) is commonly known but each community member’s valuation is her private information.

It is simple to see how all the notions we invoke in the text can be translated here. Property rights become voluntary participation: that individuals (have the option) not be forced to pay more than \( v_i \). Efficiency requires that the good be created if and only if \( V \equiv \sum_i v_i > C \). Essentially every term receives a negative sign and otherwise everything is left unchanged. The only substantial changes in the claims about the mechanisms are that the internal-to-the-community efficiency guarantees, which translate into bilateral and asymptotic efficiency, here guarantee full efficiency.

1.b Collaboration

With a slight reformulation and a few natural assumptions, our model also applies to Cournot’s original problem of collaboration among firms. Consider Cournot’s model with unknown, idiosyncratic (constant marginal) costs for each firm. Firms charge or are assigned a
price $p_i$ and may also pay a tax to the market organizer. Their payoff is $(p_i - c_i) \left( \sum_{j=1}^{N} p_j \right) - t_i$, where $t_i$ is any tax they pay.

Suppose that, rather than firms choosing their price, they are assigned a price that is their share $s_i$ of a total collective price $P$. Each collaborator $i$’s payoff is then $(s_i P - c_i) Q(P)$. If we let $C = \sum_{j=1}^{N} c_j$, then clearly a cartel planner trying to maximize industry-wide profits who know all costs would set $P$ like a monopolist with cost $C$ facing demand $Q$ according to the classic equation $MR = C$; we assume marginal revenue is strictly declining in quantity so this equation implies a unique price $P^\ast$. However, each firm’s profits are proportional to those of a linear cost monopolist with cost $\frac{C}{s_i}$. Thus if $s_i$ is larger (smaller) than $\frac{C}{C^\ast}$, firm $i$ will prefer a higher (lower) price $P$ than is in the interests of the cartel, just as a seller assigned too small of a share in the standard holdout problem above will tend to want a higher reserve price than is in the community’s interest. Thus if the cartel must assess a tax on the individual collaborator for any harm she causes other members of the cartel. This suggests the mechanism shown in the box above, with $C_i = \frac{C - c_i}{1 - s_i}$.

There are a few main differences, all minor, between collaboration and the holdout problem:

1. In the holdout problem, the quantity of sale in Cournot’s problem becomes a probability of sale. Formally let $F$ be the cumulative distribution function of $b$ and let $Q \equiv 1 - F$. Thus the two problems can only be compared if agents are risk-neutral.

2. Cournot’s assumption that demand is commonly known to all collaborators requires that, in the holdout problem, conditional on their valuation, all sellers have the same beliefs about the distribution of buyer valuations $Q$.

3. In Cournot’s model, all collaborators know all others’ opportunity cost of sales $c_i$. This trivializes the mechanism design problem, as a smart group of firms will simply agree all to price at cost in exchange for a share of total profits (Spengler, 1950). What complicates this, in the holdout problem, is that values $v_i$ (opportunity costs of sales) are private information.

4. In collaboration, all payments by buyers, $Q(P) P$, where $P \equiv \sum_i p_i$, end up in the hands of sellers. In the holdout problem, mechanisms may involve some of these revenues being lost to a third party. This means there may be some separate transfer $t_i$ from (or, if negative, to) the collaborators. Then the expected payment of each buyer is $Q(P) P$, while only $Q(P) P - \sum_i t_i$ is captured by the sellers.

5. Finally, and most importantly, collaboration accords all bargaining power to the collaborators by making them price setters, while in the holdout problem buyers need not receive a take-it-or-leave it offer.

Otherwise the two problems are identical.

**Proposition 10.** Cournot’s collaboration model where demand never exceeds $\bar{Q}$ at weakly positive prices is a special case of the holdout model of Section III where

1. All sellers are risk-neutral.
2. Sellers have common priors over buyer valuations and seller valuations are not informative about the distribution of seller valuations.

3. One only considers mechanisms where the buyer is a price-taker (the mechanism is straightforward for buyers).

Note that this provides an alternative proof of the Bergstrom’s Corollary (see the Online Appendix). Cournot’s industrial organization is just another mechanism for the holdout problem: each seller announces a reservation price and the buyer either agrees to pay the sum of all such announced prices or to forgo the purchase. Because the case of known costs can be thought of as a limit of normal distributions with increasingly small variance, we know by Mailath and Postelwaite’s Corollary (see the Online Appendix) that as the number of collaborators grows large the number of sales must dwindle to 0 under any mechanism, including Cournot’s.

**Proof.** Collaborators payoffs are by definition $Q(P)(p_i - c_i) - t_i$. Consider any holdout mechanism in which the seller is made a take-it-or-leave-it offer of $P$. The probability she accepts is $1 - F(P)$ and, in the case of acceptance each seller loses $v_i$. However, each seller also receives an expected transfer $T_i$. Thus the expected payoff of seller $i$ is $- (1 - F(P))v_i - T_i$.

If we define $v_i \equiv Qc_i$, $F(P) \equiv 1 - \frac{Q(P)}{Q}$ and $T_i \equiv Q(P)p_i - t_i$ it is clear that the payoffs of the two games are equivalent under our the proposition hypotheses.

Thus any holdout mechanism in which the buyer is a price taker immediately yields an alternative mechanism for managing collaboration. Just as with the other applications of the holdout problem, collaboration is a pervasive problem in the economy. Being one of the first mathematical economists, Cournot was not yet at the point of having to consider problems of second-order social importance! Collaboration problems arise when many firms have patents on various components of a joint product (Lerner and Tirole, 2004) and whenever one firm with substantial market power sells to another with similar power (Spengler, 1950), such as those between a union and a product market powerful employer as well as any other vertically related industry.

The extensions of the definitions of efficiency, straightforwardness and implementability to this context are obvious so we do not state them formally. The extension of property rights is *individual bankruptcy-proofness*. This is the requirement, which we abstain from defining overly formally here, that each seller can report a cost, which ensures that she pays no tax and receives a price weakly above her cost. A natural weakening of this is *approximate individual bankruptcy-proofness*, that each seller can report a cost, which ensure she pays no tax and receives a price weakly above $s_i \sum_{j \neq i} c_j / (1 - s_i)$. *Collective bankruptcy proofness* is that group of all collaborators can collude so that $P > C$ and no taxes are paid.

Of course, the justification for various design principles may be weaker in Cournot’s setting than in others. It is probably less important that the mechanism be straightforward for producers; this is helpful as most mechanisms, which make a take-it-or-leave-it offer to buyers are not straightforward for sellers. Furthermore, the justifications of protecting property rights are different: here protecting property rights means ensuring no firm makes a loss. This constraint is likely only important on average (collectively) to avoid the necessity of state subsidies or individually for those firms that cannot be coerced to produce. Nonetheless,
these differences are fairly minor and for the most part the holdout problem can reasonably be seen to encompass mechanisms for addressing Cournot’s problem of collaboration.

1.c More general public goods

Consider a public goods environment where consumers’ values for the continuous good are quasi-linear in money. Let $e$ be the total expenditure on the good. Suppose that each consumer has a value $v_i$ and the utility she gains from the public good is $v_i f(E)$, where $f$ is smooth, increasing, concave and has $f(0) = 0$. We can then write aggregate utility for expenditure level $e$ as

$$V f(E) - E = f(E) \left( V - \frac{E}{f(E)} \right).$$

Note that under our assumptions $\frac{E}{f(E)}$ is monotonically increasing in $E$ as $f(E) - f'(E) E > 0$ for all $E$. Thus the function $P(E) = \frac{E}{f(E)}$ is invertible and we can rewrite the social payoff as $f\left( P^{-1}[P] \right) (V - P)$. Suppose each consumer $i$ is asked to furnish an expenditure $e_i$ of the total expenditure. Then clearly her payoff is $f\left( P - 1 \right)(V - P) = f(E) - f'(E) E > 0$ for all $E$. Thus just as any holdout mechanism applies to binary public goods games, any collaboration mechanism applies to the environment described here.

2 Straightforward Collaboration

The Straightforward Collaboration mechanism is as described in the text except that a refund is given to each seller $i$ in the amount of

$$s_i \min \sum_{j=1}^{N} (1 - s_j) \left( \left[ P^{*} \left( \hat{C}_j \right) - \hat{C}_j \right] Q \left[ P^{*} \left( \hat{C}_j \right) \right] - \left[ P^{*} \left( \hat{C} \right) - \hat{C}_j \right] Q \left[ P^{*} \left( \hat{C} \right) \right] \right)$$

where $\hat{C}$ and $\hat{C}_j$ are calculated as before, but with $c_i$ replaced with $\hat{c}_i$. Arguments for the various properties of the VCG Collaboration mechanism are sufficiently simple as not to merit formal proof, given the development of SC in the text. Approximate individual bankruptcy-proofness follows from the fact that a collaborator submitting $\hat{c}_i = s_i \sum_{j=1}^{N} c_j (\frac{s_j - s_i}{1 - s_i})$ will never pay taxes and receives a prices $P^{*} (C_i) \geq C_i$ as optimal monopoly prices are always weakly above marginal cost. Collective bankruptcy-proofness follows from the fact that if all collaborators submit their share of $C$ none pays any taxes and $P^{*}(C) \geq C$. Profit maximization follows from the decision rule, but the monopoly distortion obviously persists. Straightforwardness is by the Groves construction of the mechanism.

3 Other collaboration mechanisms

We conclude this appendix by stating in just a bit more detail the workings of the other collaboration mechanisms described in the text.
1. APC: Each collaborator pays her full maximized profits in taxes \( s_i P^\star \left[ \frac{c_i}{s_i} \right] - c_i \) \( Q \left( P^\star \left[ \frac{c_i}{s_i} \right] \right) \). Each collaborator receives a refund of her share of the reported maximal profits of all other collaborators \( s_i \sum_{j \neq i} \frac{\left( s_j P^\star \left[ \frac{c_j}{s_j} \right] - c_j \right) Q \left( P^\star \left[ \frac{c_j}{s_j} \right] \right)}{1-s_j} \).

2. FPC: Each collaborator pays in tax the surplus she gains by the impact she has on the price \( s_i P^\star (C) - c_i \) \( Q \left( P^\star (C) \right) \) and is refunded her share of all other seller’s surplus \( s_i \sum_{j \neq i} \frac{\left[ s_j P^\star (C) - c_j \right] Q \left( P^\star (C) \right) - \left[ s_j P^\star (C_j) - c_j \right] Q \left( P^\star (C_j) \right)}{1-s_j} \).

3. X-plurality: Each seller announces a cost \( c_i \) and this is transformed into a collective cost \( \frac{c_i}{s_i} \). The \( X \)-th quantile collective cost is determined by ranking all \( \frac{c_i}{s_i} \) and choosing those lowest value so that the sum over all \( i \) below this of \( s_i \) is at least \( X \). We refer to this quantity as \( C_X (c, s) \). A price \( P^\star (C_X [c, s]) \) is charged to consumers and each seller is given a share \( s_i \) of the revenue. No other money changes hands. This is equivalent (except over regions of increasing marginal revenue, which all sellers will be unanimous in wanting to bipass) to trying out many different price levels and at each asking each for each collaborator to vote in favor of raising or lowering the price, then charging the minimum price at which at least \( X \) of the collaborators support a lower price.