Example of Vickey-Clark-Groves Mechanism

Let us try the following principle. Suppose everyone has quasi-linear utility. We ask everybody to report his preferences. The government takes the action that maximizes the sum of reported utilities. The government makes each individual $i$ pay an amount equal to the difference in the summed value to the rest of the population between the outcome they get and the outcome that they would get if no attention was paid to $i$’s preferences.

Remember our discussion of the pivotal mechanism? The government chooses the outcome that maximizes the sum of reported willingness to pay and collects nothing from an individual $i$ if his report doesn’t change the outcome. If it does shift the outcome, then $i$ pays an amount equal to the difference between amounts of the two sides not counting his own report. This amount is exactly the difference $X - Y$ between the summed value $X$ to the rest of the population of the outcome when $i$’s answer is not counted and the summed value $Y$ to the rest of the population if $i$’s value is counted.

This same principle extends to cases where the government’s choice can be any positive number rather than just a binary choice. A more general form of this problem. There are $n$ people who share consumption of a public good. Each person has a quasilinear utility function $U(x, y) = x + u_i(y)$ and where $u_i$ is a single-peaked function that increases for $y \leq y_i^+$ and decreases for $y \geq y_i^+$.

Step 0: All players are asked to report their utility functions to the authority. Let $m_i(\cdot)$ be the function reported by $i$.

Step 1: Where the vector of reported utility functions is given by $m$, the central authority chooses $\bar{y}(m)$ to maximize the sum of the reported utility functions.

Step 2: The central authority makes a sidepayment to each $i$ that is equal to the sum of the reported utilities of $\bar{y}$ for the other $n - 1$ persons. This is $Y(m) = \sum_{j \neq i} m_j(\bar{y}(m))$.

Step 3: The central authority collects from each person, an amount $Z(m)$ equal to the maximum possible sum of the reported utilities of the other persons:

$$Z(m) = \max_y \sum_{j \neq i} m_j(\bar{y})$$

Note the following:

- The amount $Z(m)$ defined in Step 3 does not depend on $i$’s reported utility function.

- Since $Z(m) = \max_y \sum_{j \neq i} m_j(\bar{y})$, it must be that the $Z(m) \geq \sum_{j \neq i} m_j(\bar{y}) = Y(m)$. It follows that the net tax paid by person $i$, which is $Z(m) - Y(m)$ must be positive. Interpreting this net tax, we see that it is the effect of person $i$’s response on the total payoffs of the other players.

- If the vector of messages sent by all players is $m$, and the government chooses $y$, then the payoff to player $i$ will

$$x_i + u_i(y) + \sum_{j \neq i} m_j(y) - Z.$$
Recall that the government chooses $y$ to maximize

$$m_i(y) + \sum_{j \neq i} m_j(y).$$

What is the best thing that player $i$ can do for himself? If he reports $m = u$, then the government will choose $\bar{y}$ to maximize

$$x_i + u_i(y) + \sum_{j \neq i} m_j(y) - Z.$$

If $i$ reports $m \neq u$, the government will solve a different maximization problem with a lower value of

$$x_i + u_i(y) + \sum_{j \neq i} m_j(y) - Z.$$

With these results we conclude

1. reporting one’s true utility is a weakly dominant strategy.
2. When everybody uses their weakly dominant strategy, the outcome is the Pareto optimal amount of $y$.
3. The net tax collected from each person is non-negative.
4. If a positive taxes are collected, the outcome is not efficient, since the mechanism doesn’t allow for them to be rebated.

This procedure is more general than may first appear. What if the public good costs money? Add the following wrinkle. Assign tax shares $1/n$ (or any other division of shares that adds to 1). If the government chooses $y$ then $i$ pays a tax of $p_y y/n$. This gives us single-peaked utility functions for $y$ and we can just repeat the above exercise with the reinterpreted utilities.

**An illustrative special case:**

Three friends, Archie, Betty, and Veronica are planning a party. They disagree about how many people to invite. Each person $i$ has an initial endowment of $W_i$ dollars and a quasilinear utility function of the form

$$u_i(x_i, y) = x_i + a_i y - \frac{1}{2} y^2$$

where $x_i$ is the number of dollars that $i$ has to spend and $y$ is the number of people invited to the party. Each of them knows the functional form of their utility functions, but only person $i$ knows the parameter $a_i$ from his or her own utility function.
Each person is asked to report his or her parameter \( a_i \). They are not necessarily required to tell the truth. Let \( m_i \) be the value reported by person \( i \).

Suppose that \( a_A = 20 \), \( a_B = 40 \), and \( a_C = 60 \). If each person plays his or her best strategy, then \( m_i = a_i \) for all \( i \). In this case, the number of persons invited is \( 40 = \frac{a_A + a_B + a_C}{3} \). Archie receives a sidepayment equal to \( (m_B + m_V)40 - 40^2 = 2400 \) and makes a payment of \( (m_B + m_V)^2/4 = 2500 \), so Archie’s net payment is 100.

Betty receives a sidepayment equal to \( (m_A + m_V)40 - 40^2 = 1600 \) and makes a payment of \( (m_A + m_V)^2/4 = 1500 \) and so makes a net payment of 0.

Veronica receives a sidepayment equal to \( (m_A + m_B)40 - 40^2 = 800 \) and makes a payment of \( (m_A + m_B)^2/4 = 900 \), so she makes a net payment of 100.

Thus the mechanism wastes 200.

A more general quadratic example

**Step 1:** Where the vector of reported utility functions is given by \( m \), the central authority chooses \( y \) to maximize the sum of the reported utility functions. That is, it chooses \( y \) to maximize

\[
\sum_{i=1}^{n} \left( a_i y - \frac{1}{2} y^2 \right) = y \sum_{i=1}^{n} a_i - \frac{n}{2} y^2. \quad (2)
\]

Setting the derivative with respect to \( y \) equal to zero, we see that this sum is maximized when \( y = y(m) \) where

\[
y(m) = \frac{1}{n} \sum_{i} a_i. \quad (3)
\]

**Step 2:** The central authority makes a sidepayment to each \( i \) that is equal to the sum of the reported utilities of \( y \) for the other two persons. For any person \( i \), this means that person \( i \) gets a sidepayment equal

\[
y \sum_{j \neq i} m_j - \frac{n-1}{2} y^2 \quad (4)
\]

**Step 3:** The central authority collects from each person, an amount equal to the maximum possible sum of the reported utilities of the other persons. For any person \( i \), this amount is equal to

\[
\max_y \{ y \sum_{j \neq i} m_j - \left( \frac{n-1}{2} \right) y^2 \}. \quad (5)
\]

Setting the derivative equal to zero, we see that this sum maximized when

\[
y = \frac{\sum_{j \neq i} m_j}{n-1}.
\]
and hence the maximal possible sum of utilities for the other persons is

\[ \frac{1}{2(N-1)} \sum_{j \neq i} m_j^2. \]

**Solving Person \( i \)'s decision problem:**

Where the number of persons attending the party determined by the mechanism, given the vector of responses \( m \), is \( y(m) \), when we take account of the payments in steps 2 and 3, Person \( i \) will have consumption \( x_i(m) \) of other goods where

\[ x_i(m) = W_i + y(m) \sum_{j \neq i} m_j - \frac{n-1}{2} y(m)^2 - \frac{1}{2(N-1)} \sum_{j \neq i} m_j^2. \]  

(6)

Given that \( i \)'s utility function is \( u_i(x_i(m), y(m)) = x_i(m) + a_i y(m) - \frac{1}{2} y(m)^2 \), we can calculate \( i \)'s utility. In particular to maximize his utility, \( i \) will choose \( m_i \) that maximizes this utility, which is equal to

\[ u_i(x_i(m), y(m)) = x_i(m) + a_i y(m) - \frac{1}{2} y(m)^2. \]  

(7)

To find the response \( m_i \) that maximizes his utility, \( i \) would set the partial derivative of Equation 7 with respect to \( m_i \) equal to 0. This implies that

\[ \frac{\partial x_i(m)}{\partial m_i} + a_i \frac{\partial y(m)}{\partial m_i} - y(m) \frac{\partial y(m)}{\partial m_i} = 0. \]  

(8)

From Equation 3 we have

\[ \frac{\partial y(m)}{\partial m_i} = \frac{1}{n}. \]  

(9)

From Equations 6 and 9 it follows that

\[ \frac{\partial x_i(m)}{\partial m_i} = \left( \sum_{j \neq i} m_j \right) \frac{\partial y(m)}{\partial m_i} - (n - 1) y(m) \frac{\partial y(m)}{\partial m_i} \]

\[ = \frac{1}{n} \sum_{j \neq i} m_j - \frac{n-1}{n} y(m). \]  

(10)

Then from substituting from 10 and 9 into 8 we have

\[ \frac{1}{n} \left( a_i + \sum_{j \neq i} m_j \right) - y(m) = 0 \]  

(11)

It follows from Equation 3 and 11 that

\[ \frac{1}{n} \left( a_i + \sum_{j \neq i} m_j \right) - \sum_{j=1}^{n} m_j = 0 \]  

(12)
which implies that \( m_i = a_i \). Thus we have demonstrated that whatever numbers
the others claim describe their utility functions, the best response for person \( i \)
is to announce his true value \( m_i = a_i \).

Since this reasoning applies for every individual \( i \), it must be that announc-
ing \( m_i = a_i \) is a dominant strategy for every individual. Hence if all play
their dominant strategies, the quantity of \( y \) selected will be the Pareto efficient
quantity \( \frac{1}{n} \sum_{i=1}^{n} a_i \) that maximizes the sum of utilities.

**Back to Archie, Betty, and Veronica:**

### 0.0.1 Groves-Ledyard Mechanism

Let us try applying the Groves-Ledyard mechanism to the same \( n \) person society.

In the Groves Ledyard mechanism, each player reports a number \( m_i \) and the
size chosen for the party is the sum of these numbers. Define the mean of the
reported numbers to be

\[
\mu = \frac{1}{N} \sum_{j=1}^{n} m_j
\]

and define

\[
\mu_i = \left( \frac{1}{N - 1} \right) \sum_{j \neq i} m_j
\]

to be the mean of the numbers named by persons other than \( i \). Define \( \sigma_i^2 \) to be
the variance of the numbers submitted by persons other than \( i \). If the vector of
numbers submitted is \( m \), each player \( i \) will pay a net tax

\[
T_i(m) = \frac{\gamma}{2} \left( \frac{N - 1}{N} (m_i - \mu_i)^2 - \sigma_i^2 \right).
\]

With this tax scheme and with the size of party being \( x = \sum_i m_i \), the utility
of person \( i \) will be

\[
W_i + a_i \sum_i m_i - \left( \sum_i m_i \right)^2 - T_i(m)
\]  \tag{13}

Person \( i \) will find his or her best choice of \( m_i \) by setting the derivative with
respect to \( m_i \) of Expression 13 equal to zero. This happens when

\[
a_i - \sum m_i - \frac{\partial T_i(m)}{\partial m_i} = 0
\]

or equivalently when

\[
a_i - \sum m_i = \gamma \left( \frac{N - 1}{N} (m_i - \mu_i) \right)
\]  \tag{14}
A bit of simple algebra\(^1\) shows that

\[
\frac{N - 1}{N} (m_i - \mu_i) = m_i - \mu
\]

and therefore Equation 14 is equivalent to

\[
a_i - \sum m_i = \gamma (m_i - \mu) \tag{15}
\]

Summing both sides of Equation 15 over \(n\), we find that

\[
\sum_{i=1}^{n} a_i - n \sum_{i=1}^{n} m_i = 0 \tag{16}
\]

and therefore

\[
\sum_{i=1}^{n} m_i = \frac{1}{n} \sum a_i \tag{17}
\]

---

\(^1\)Proof is as follows:

\[
m_i - \mu = m_i - \frac{1}{N} \left( \sum_{j=1}^{N} m_j \right)
\]

\[
= m_i \left( 1 - \frac{1}{N} \right) - \frac{1}{N} \sum_{j \neq i} m_j
\]

\[
= m_i \frac{N - 1}{N} - \frac{N - 1}{N} \mu_i
\]

\[
= \frac{N - 1}{N} (m_i - \mu_i)
\]