Chapter 11

The Groves-Clarke Mechanism

Let there be one private good and one public good. Consumer i has the utility function

$$U_i(X_i, Y) = X_i + F_i(Y) \tag{11.1}$$

where X_i is his private good consumption and Y is the amount of public good. Each i has an initial endowment of W_i units of private good. Public good must be produced using private goods as an input. The total amount of private goods needed to produce Y units of public good is a function $C(Y)$. Assume that F_i is a strictly concave function and C a convex function. If we consider only allocations in which everyone receives at least some private good, then for this economy there is a unique Pareto optimal quantity of public good. This quantity maximizes

$$\sum_i F_i(Y) - C(Y) \tag{11.2}$$

Consumers are asked to reveal their functions F_i to the government. Let M_i (possibly different from F_i) be the function that consumer i claims. Let $M = (M_1, \cdots, M_n)$ be the vector of functions claimed by the population. If the reported vector is M, the government chooses an amount of public goods, $Y(M)$, such that

$$\sum_i M_i(Y(M)) - C(Y(M)) \geq \sum_i M_i(Y) - C(Y) \tag{11.3}$$

for all $Y \geq 0$.

Taxes $T_i(M)$ are then assigned to each consumer i where

$$T_i(M) = C(Y(M)) - \sum_{j \neq i} M_j(Y(M)) - R_i(M) \tag{11.4}$$

and where $R_i(M)$ is some function that may depend on the functions, M_j, reported by consumers other than i but is constant with respect to M_i.

If the vector of functions reported to the government is $M = (M_1, \cdots, M_n)$, then Consumer i's private consumption is

$$X_i(M) = W_i - T_i(M) \tag{11.5}$$

and his utility is

$$X_i(M) + F_i(Y(M) = W_i + \sum_{j \neq i} M_j(Y(M) + F_i(Y(M) - C(Y(M) + R_i(M) \tag{11.6}$$

Since $W_i + R_i(M)$ is independent of M_i, we notice that the only way in which i's stated function M_i affects his utility is through the dependence of $Y(M)$ on M_i.

We see, therefore from 11.6 that given any choice of strategies by the other players, the best choice of M_i for i is the one that leads the government to choose $Y(M)$ so as to maximize

$$\sum_{j \neq i} M_j(Y) + F_i(Y) - C(Y). \tag{11.7}$$

But recall from expression 11.3 that the government attempts to maximize

$$\sum_{j=1}^{n} M_j(Y) - C(Y). \tag{11.8}$$

Therefore if consumer i reports his true function, so that, $M_i = F_i$, then when the government is maximizing 11.8 it maximize 11.7. It follows that the consumer can not do better and could do worse than to report the truth. Honest revelation is therefore a dominant strategy.

Since everyone chooses his dominant strategy, true preferences are revealed and the government's choice of $Y(M)$ is the value of Y that maximizes

$$\sum_{j=1}^{n} F_j(Y) - C(Y) \tag{11.9}$$

This leads to the correct amount of public goods. Of course for the device to be feasible, it must be that total taxes collected are at least as large as the total cost of the public goods. If the outcome is to be Pareto optimal, the amount of taxes collected must be no greater than the total cost of public goods. Otherwise private goods are wasted. We are left, therefore, with the task of trying to rig the functions $R_i(M)$ in such a way to establish this balance. In general, it turns out to be impossible to find functions $R_i(M)$ that are independent of M_i for each i and such that

$$\sum_i T_i(M) = C(Y(M)) \tag{11.10}$$

However, Clarke and also Groves and Loeb found functions $R_i(M)$ that guarantee that tax revenues at least cover total costs.

Their idea can be explained as follows. Suppose that for each i, the government sets a "target share" $\theta_i \geq 0$ where $\sum_i \theta_i = 1$. The government then tries to fix $R_i(M)$ so that $T_i(M) \geq \theta_i C(Y(M))$ for each i. Then, of course, $\sum_i T_i(M) \geq C(Y(M))$. From equation (3), it follows that

$$T_i(M) - \theta_i C(Y(M)) = [(1-\theta_i)C(Y(M)) - \sum_{j \neq i} M_j(Y(M))] - R_i(M). \tag{11.11}$$

Therefore the government could set $T_i(M) = \theta_i C(Y(M))$ if and only if it could set

$$R_i(M) = (1 - \theta_i)C(Y(M)) - \sum_{j \neq i} M_j(Y(M)). \tag{11.12}$$

But in general such a choice of $R_i(M)$ would be inadmissible for our purpose because $R_i(M)$ depends on M_i, since $Y(M)$ depends on M_i.

Suppose that the government sets

$$R_i(M) = \min_Y [(1 - \theta_i)C(Y) - \sum_{j \neq i} M_j(Y)]. \tag{11.13}$$

Then $R_i(M)$ depends on the M_j's for $j \neq i$ but is independent of M_i. From (10) it follows that with this choice of $R_i(M)$ we have:

$$T_i(M) - \theta_i C(Y(M)) \geq 0 \text{ for all } i \tag{11.14}$$

Therefore

$$\sum_i T_i(M) \geq C(Y(M)). \tag{11.15}$$

This establishes the claim we made for the Clarke tax.

Chapter 12

The Groves-Ledyard Mechanism

Groves and Ledyard propose a demand revealing mechanism which they call "An Optimal Government". The mechanism formulates rules of a game in which the amount of public goods and the distribution of taxes is determined by the government as a result of messages which the citizens choose to communicate. Although the government has no independent knowledge of preferences, and citizens are aware that sending deceptive signals might possibly be beneficial, it turns out that Nash equilibrium for this game is Pareto optimal. The Groves–Ledyard mechanism is defined for general equilibrium and applies to arbitrary smooth convex preferences.

In contrast, the Clarke tax (discovered independently by Clarke [1971] and Groves and Loeb [1975]) is well defined only for economies in which relative prices are exogenously determined and where utility of all consumers takes the quasi–linear form:

$$U_i(X_i, Y) = X_i + F_i(Y). \tag{12.1}$$

The Clarke tax has the advantage that for each consumer, equilibrium is a dominant strategy equilibrium rather than just a Nash equilibrium. Thus there are no complications related to stability or multiple equilibria. On the other hand, the Clarke tax has the disadvantages that although it leads to a Pareto efficient amount of public goods it generally will lead to some waste of private goods.

Suppose that there are n consumers, and one public good and one private good. Each consumer has an initial endowment of W_i units of private good. Public good is produced at a constant unit cost of q.

The government asks each consumer i to submit a number, (positive or negative) m_i. The government will supply an amount of public goods $Y = \sum_i m_i$. To describe the Groves-Ledyard mechanism efficiently it is useful to define the following bits of notation: Define

$$\bar{m}_{\text{-}i} = \frac{1}{n-1} \sum_{j \neq i} m_j \tag{12.2}$$

to be the average of the numbers submitted by persons other than i. We will also define a function

$$R_i(m) = \frac{1}{n-2} \sum_{j \neq i} (m_j - \bar{m}_{\text{-}i})^2 \tag{12.3}$$

For the time being the main thing that we should notice about the odd-looking expression 12.3 is that $R_i(m)$ depends on the m_j's for $j \neq i$, but does not depend on m_j. As we will see, we will use these expressions to make budgets balance.

When the vector of messages sent by individuals is $m = (m_1, \ldots, m_n)$, the Groves-Ledyard mechanism will impose a tax on individual i equal to

$$T^i(m) = \alpha_i q \sum_{k=1}^{n} m_k + \frac{\gamma}{2} \left(\frac{n}{n-1} (m_i - \bar{m}_{\text{-}i})^2 - R_i(m) \right) \tag{12.4}$$

where the α_i's and γ are arbitrarily chosen positive parameters and $\sum_k \alpha_k = 1$. (Though Expression 12.4 looks nasty, remember that it is only a quadratic, and we are soon going to defang this beast by differentiating it.)

If the vector of messages is $m = (m_1, \cdots, m_n)$, consumer i's utility will be

$$W_i - T^i(m) + F_i\left(\sum_{k=1}^{n} m_k\right). \tag{12.5}$$

Therefore in a Nash equilibrium each consumer i would be choosing m_i to maximize 12.5. The first order condition for maximizing 12.5, is the relatively mild-appearing expression:

$$F_i'\left(\sum_k m_k\right) = \gamma\left[m_i - \frac{1}{n} \sum_k m_k\right] + \alpha_i q \tag{12.6}$$

Summing the equations in 12.6 and recalling that $\sum_k \alpha_k = 1$, we see that

$$\sum_k F_k'\left(\sum_k m_k\right) = q. \tag{12.7}$$

This is the Samuelson condition for efficient provision of public goods.

The trickiest thing to show is that total revenue collected by the Groves-Ledyard tax equals the total costs of the public good. To find this out, we sum the taxes collected from each i to find that

$$\sum_{i=1}^{n} T_i(m) = \sum_{i=1}^{n} \alpha_i q \sum_{k=1}^{n} m_k + \frac{\gamma}{2} \sum_{i=1}^{n} \left(\frac{n}{n-1}(m_i - \bar{m}_{\sim i})^2 - R_i(m) \right) \quad (12.8)$$

Some fiddling with sums of quadratics will give us the result that

$$\sum_{i=1}^{n} \frac{n}{n-1}(m_i - \bar{m}_{\sim i})^2 = \sum_{i=1}^{n} R_i(m) \quad (12.9)$$

Therefore Equation 12.8 simplifies to:

$$\sum_{i=1}^{n} T_i(m) = \sum_{i=1}^{n} \alpha_i q \sum_{k=1}^{n} m_k \quad (12.10)$$

Since $\sum_{k=1}^{n} m_k = Y$,and $\sum_{i=1}^{n} \alpha_i = 1$, this expression simplifies further to

$$\sum_{i=1}^{n} T_i(m) = qY \quad (12.11)$$

which menas that revenue exactly covers the cost of the public good.

The Groves-Ledyard Mechanism with Quasi-linear Utility

It is of some interest to examine the nature of the Groves–Ledyard mechanism as applied to the case of quasi-linear utility, where each consumer i has a utility function $U_i(X_i, Y) = X_i + F_i(Y)$. Studying the quasilinear case will help us to develop some "feel" for the device by seeing how it performs in a manageable environment. It also is useful to compare the merits of this system with the Groves-Clarke mechanism when both are operating on Groves-Clarke's home turf. (remember that the Groves-Clarke mecahnism is defined *only* for quasilinear utility.)

We are able to show quite generally that when there is quasi–linear utility, the Groves–Ledyard mechanism has exactly one Nash equilibrium.

Furthermore, this equilibrium is quite easily computed and described. This is of some interest because, in general, little is known about the uniqueness of Groves–Ledyard's equilibrium and the question of the existence of equilibrium is also less than satisfactorily resolved.

Since $F_k'' < 0$, Equation 12 has a unique solution for $\sum_k m_k$. Let $\bar{Y}$ denote this solution. Now define

$$\beta_i = F_i'(\bar{Y}). \tag{12.12}$$

Then 12.6 can be rewritten as

$$\beta_i = \gamma[m_i - \frac{1}{n}\bar{Y}] + \alpha_i q. \tag{12.13}$$

Now α_i, q and γ are parameters and β_i is uniquely solved for by and 12.12. Thus we solve uniquely for m_i as follows:

$$m_i = \frac{1}{\gamma}(\beta_i - \alpha_i q) + \frac{\bar{Y}}{n}. \tag{12.14}$$

This establishes our claim that in the case of quasi–linear utility, Nash equilibrium exists, is unique and is easily computed.