Solutions for Public Finance problem sets

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Chapter 2: Public goods and Private goods

Exercise 2.1

(a) The equation implied by the Samuelson condition requires that:

$$\sum_{i=1}^{1000} MRS_i = \text{marginal cost of public good}$$

The marginal rate of substitution between skating rink and Labatt’s ale for a typical citizen:

$$MRS_i = \frac{\partial U_i(X_i, Y)}{\partial Y} \cdot \frac{\partial Y}{\partial U_i(X_i, Y)} = \frac{100}{Y^2}$$

So, the Samuelson condition is $$\frac{100000}{Y^2} = 10$$.

(b) Since the MRS_i depends only on the public good, the left side of the Samuelson condition will also depend only on the public good. So, this equation uniquely determines the efficient rink size for Muskrat because:

(i) $$\frac{\partial U_i^2(X_i, Y)}{\partial^2 Y} = -\frac{200}{Y^3} < 0$$;

(ii) left side of Samuelson condition is a decreasing function of Y.

Given the marginal cost of the public good, there can be at most one value of Y that satisfies that equation.

The Pareto optimal amount of skating rink size is the [Y=100].
Exercise 2.2

Since we have quasilinear utility functions, the sum of the MRS$_i$ will depend only on the fireworks. So, we do not need to use the feasibility condition to determine Pareto optimal amount of fireworks for Cowflop.

The Samuelson condition is:

$$\sum_{i=1}^{1100} MRS_i = \sum_{i=1}^{1100} \frac{\partial U_i(X_i, Y)}{\partial X_i} = \frac{P_Y}{P_X}$$

$$MRS_i = \frac{a_i}{2000 \sqrt{Y}}$$

and note that $$\sum_{i=1}^{1000} i = 1 + 2 + 3 + \ldots + 1000 = \frac{1000 \times 1001}{2}.$$

So, we have $$\sum_{i=1}^{1000} \left( \frac{i}{10 \times 2000 \sqrt{Y}} \right) = 1$$ and $$Y = 626.251.$$

Exercise 2.3

(a) In order to find the optimal amount of public good as a function of the parameters of the problem we have to solve the following:

Max$_{\left\{ x_1, \ldots, x_{1000} \right\}} U^L_1 = x_1^\alpha Y^{1-\alpha}$

\[
\begin{align*}
&\text{s.t.} \\
&\quad x_1^\alpha Y^{1-\alpha} \geq U_i^L, \quad i = 2, \ldots, 500 \\
&\quad x_j^\beta Y^{1-\beta} \geq U_j^0, \quad j = 1, \ldots, 500 \\
&\quad \sum_{i=1}^{500} x_i + \sum_{j=1}^{500} x_j + Y \leq \sum_{i=1}^{500} W_i + \sum_{j=1}^{500} W_j = w
\end{align*}
\]

From the resolution of this problem we get:
\[
\begin{align*}
Y = \frac{1 - \alpha}{\alpha} \sum_{i=1}^{500} x_i + \frac{1 - \beta}{\beta} \sum_{j=1}^{500} x_j & \quad \text{(Samuelson Condition)} \\
\sum_{i=1}^{500} x_i + \sum_{j=1}^{500} x_j + Y &= w \quad \text{(feasibility condition)}
\end{align*}
\]

From this system we get:

\[
Y = (1 - \alpha)W + \alpha \sum_{j=1}^{500} x_j \left( \frac{1 - \beta}{\beta} - \frac{1 - \alpha}{\alpha} \right)
\]

As we see, the optimal amount of public good depends on:

(i) aggregate income of the community;
(ii) on how the private goods are divided between the Larsons and the Olsens;
(iii) on the shares \( \alpha \) and \( \beta \).

(b) If \( \alpha = \beta \), then all individuals have identical Cobb-Douglas utility functions.

The optimal amount of public good, in this case, is reduced to:

\[
Y = (1 - \alpha)W
\]

and it depends only on the aggregate income but not on how that income is distributed. Thus, any redistribution of income that leaves the aggregate income in the community unchanged will have no effect on the efficient allocation of \( Y \).

An example would be:

Suppose the two following income distribution:

- \( \left( w_1^L, w_2^L, w_3^L, \ldots, w_{500}^L; w_1^O, w_2^O, \ldots, w_{500}^O \right) \)
- \( \left( \bar{w}_1^L, \bar{w}_2^L, \bar{w}_3^L, \ldots, \bar{w}_{500}^L; \bar{w}_1^O, \bar{w}_2^O, \bar{w}_3^O, \ldots, \bar{w}_{500}^O \right) \)

where, \( w_1^L = w_1^L - \delta \) and \( w_2^L = w_3^L + \delta \),
\( \bar{w}_2^L = w_2^O + \delta \) and \( \bar{w}_3^L = w_3^O - \delta \).
It is easy to see that both will conduct to the same optimal amount of $Y$ since the sum of all individuals’ income is the same in both cases, that is, the aggregate income is unchanged.

**Exercise 2.4**

(a)

Maximize $X_C + 2\sqrt{Y}$

subject to $X_D + \sqrt{Y} \geq U_D$

$X_C + X_D + Y \leq W$

The Samuelson condition is:

$$MRS_{X,Y}^C + MRS_{X,Y}^D = 1 \iff Y^{-\frac{1}{2}} + Y^{-\frac{1}{2}} = 1$$

So, the Pareto optimal amount of $Y$ if both persons are to have positive consumption of private goods is:

$$Y = \frac{9}{4}$$

And $X_C + X_D = W - \frac{9}{4}$ \implies any allocation $(X_C, X_D)$ that divides $W - \frac{9}{4}$ units of private goods between person C and person D in some proportions can be achieved.

(b)

We want the set of Pareto optimal allocations in which one or the other person consumes no private goods:

$$(X_C, X_D, Y)$$
• At allocation \((W - \frac{9}{4}, 0, \frac{9}{4})\), person C gets all the private goods.

\[U^C = W + \frac{9}{4} \quad \text{and} \quad U^D = \frac{3}{2}\]

The allocation \((W - \frac{9}{4}, 0, \frac{9}{4})\) is Pareto optimal and the Samuelson conditions apply (since \(Y = \frac{9}{4}\) is the solution of equation \(MRS^C_{X,Y} + MRS^D_{X,Y} = 1\)).

• At allocation \((0, W - \frac{9}{4}, \frac{9}{4})\), person D gets all the private goods.

\[U^C = 3 \quad \text{and} \quad U^D = W - \frac{3}{4}\]

Allocation \((W - \frac{9}{4}, 0, \frac{9}{4})\) also is Pareto optimal and satisfies Samuelson condition.

But we can find other Pareto optimal points in which one or the other person consumes no private goods by making people act in a purely selfish way. These points, by contrast with \((W - \frac{9}{4}, 0, \frac{9}{4})\) and \((W - \frac{9}{4}, 0, \frac{9}{4})\), do not maximize the sum of utilities.

• Person C has no interest in person’s D consumption,

\[
\begin{align*}
\text{Max} & \quad X_C + 2\sqrt{Y} \\
\text{s.t.} & \quad \{X_C + Y \leq W \}
\end{align*}
\]

The solution of this problem is \(Y = 1\). So, the allocation that we have is \((w - 1, 0, 1)\) and the utilities associated to it are:

\[U^C = w + 1 > w + \frac{3}{4} \quad \text{and} \quad U^D = 1 < \frac{3}{2}\]
We have a set of Pareto optimal allocations where the utility distributions result from allocations \((W-Y, 0, Y)\) with \(Y \in \left[1, \frac{9}{4}\right]\). But for \(Y \in \left[1, \frac{9}{4}\right]\), the allocations do not satisfy the Samuelson conditions.

We know that \(\text{MRS}^C = Y^{-\frac{1}{2}}\) and \(\text{MRS}^D = \frac{1}{2} Y^{-\frac{1}{2}}\). So, take for example:

- \(Y = 1\) \(\text{MRS}^D + \text{MRS}^C = \frac{1}{2} + 1 = \frac{3}{2} > 1 = \frac{P_y}{P_x}\).
- \(Y = \frac{3}{2}\) \(\text{MRS}^D + \text{MRS}^C = 0.41 + 0.82 = 1.23 > 1\)

- Person D has no interest in person’s C consumption and acts in a selfish way.

\[
\text{Max} \quad X_{d^+} \sqrt{Y} \\
\text{s.t.} \quad X_D + Y \leq W
\]

The solution of this problem is \(Y = \frac{1}{4}\). The allocation we have now is \((0, W - \frac{1}{4}, \frac{1}{4})\) and the utilities are \(U^C = 1 < 3\) and \(U^D = W + \frac{1}{4} > w - \frac{3}{4}\).

By the same reason used in the case before, we have a set of Pareto optimal allocations in which Person C gets no private goods and where the utility distribution result from allocations of the type \((0, W-Y, Y)\), with \(Y \in \left[1, \frac{9}{4}\right]\). As before, the allocations \((0, W-Y, Y)\) with \(Y \in \left[1, \frac{9}{4}\right]\) do not meet Samuelson condition.

But for all these points be pareto optimal points is necessary to impose a restriction to \(W\).

\(W\) has to be superior to 2.25, as we will see in the next question.

*In summary,*

The set of Pareto optimal allocations in which one or the other person consumes no private goods is:

\[
S = \left\{(X_C, X_D, Y) : \left(\frac{w - \frac{9}{4}}{4}, 0, \frac{9}{4}\right) ; \left(0, \frac{9}{4}, \frac{w - \frac{9}{4}}{4}\right) ; \left(w - Y, 0, Y\right) \land Y \in \left[1, \frac{9}{4}\right] \land W > 2.25 ; \left(0, w - Y, Y\right) \land Y \in \left[1, \frac{9}{4}\right] \land W > 2.25 \right\}
\]
All these points are Pareto optimal since we cannot improve one person’s utility without making the other person worse off. On the other hand as it was shown, at these Pareto optimal points the Samuelson condition does not necessarily apply.

(c) Before look at the graph take a look in the following two tables:

(i) Consider the case where \((W-Y, 0, Y)\) with \(Y \in \left[ 1, \frac{9}{4} \right] \).

<table>
<thead>
<tr>
<th>(Y)</th>
<th>(U_C)</th>
<th>(U_D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(W+1)</td>
<td>1</td>
</tr>
<tr>
<td>1.5</td>
<td>(W+0.95)</td>
<td>1.2</td>
</tr>
<tr>
<td>2</td>
<td>(W+0.83)</td>
<td>1.4</td>
</tr>
<tr>
<td>2.25</td>
<td>(W+0.75)</td>
<td>1.5</td>
</tr>
</tbody>
</table>

(ii) Consider the case where \((0, W-Y, Y)\) with \(Y \in \left[ \frac{1}{4}, \frac{9}{4} \right] \).

<table>
<thead>
<tr>
<th>(Y)</th>
<th>(U_C)</th>
<th>(U_D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1</td>
<td>(W+0.25)</td>
</tr>
<tr>
<td>0.5</td>
<td>1.41</td>
<td>(W+0.21)</td>
</tr>
<tr>
<td>0.75</td>
<td>1.73</td>
<td>(W+0.12)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(W)</td>
</tr>
<tr>
<td>1.5</td>
<td>2.45</td>
<td>(W-0.28)</td>
</tr>
<tr>
<td>2</td>
<td>2.83</td>
<td>(W-0.59)</td>
</tr>
<tr>
<td>2.25</td>
<td>3</td>
<td>(W-0.75)</td>
</tr>
</tbody>
</table>

In order for all these points \((W-Y, 0, Y)\) be better than points \((0, W-Y, Y)\) for person C we need to assure that \(w + 0.75 > 3 \iff w > 2.25\).
By the same logic, for the points \((0, W-Y, Y)\) be better than points \((W-Y, 0, Y)\) for person D we need to assure that \(w > 1.25\).

If we consider both restrictions then we need to assure that the aggregate income is bigger than 2.25 in order for those allocations be Pareto Optimal.

The graph for an aggregate income \(W=3\) is then:

The *utility possibility frontier (UPF)* is the set \([CBAD]\). Any point between the straight line \([AB]\) can be achieved by supplying \(\frac{9}{4}\) units of public good and dividing \((w-\frac{9}{4})\) units of private goods between C and D in some proportions. The curved line \([CB]\) corresponds to the utility distributions that result from allocations \((0, W-Y, Y)\) with \(Y \in \left[\frac{1}{4}, \frac{9}{4}\right]\) and \(w = 3\). The curved line \([AD]\)
corresponds to the utility distributions that result from allocations \((W-Y, 0, Y)\) with \(Y \in \left[1, \frac{9}{4}\right]\) and \(w = 3\).

The curved segment \([DE]\) does not belong to the UPF since it consists of the utility distributions for the allocations where person C has all the private goods but the amount of public good is less than the amount person C would prefer to supply for himself. Symmetrically, the curved segment \([CF]\) also does not belong to the UPF since it consists of the utility distributions corresponding to allocations where person D gets all the private goods and the amount of public good is inferior to the amount person d would like to supply for himself.

(d) The straight-line portion of the UPF is the zone achievable with positive private consumption for both people.

At the UPF: \(U_C + U_D = X_C + 2\sqrt{Y^*} + X_D + \sqrt{Y^*}\) with \(Y^* = \frac{9}{4}\). So we have that \(U_C + U_D = X_C + X_D + \frac{9}{2}\). Since the amount of income left for C and D after the public good has been paid is \(w - \frac{9}{4}\) then we have \(U_D = w + \frac{9}{4} - U_C\).

So, the straight-line portion of UPF is: \(\{(U_D, U_C) \in \left[3, 3.75\right]: U_D = w + \frac{9}{4} - U_C\}\).

(e)

For the curved portion (CB) we have that \(X_C = 0\) and \(X_D = W - Y\). So \(U_C = 2\sqrt{Y}\) and \(U_D = W - Y + \sqrt{Y}\). From person C utility function we have that \(Y = \frac{U_C^2}{4}\) and plugging this expression into person D utility function we describe the top curved part of UPF as being \(U_D = W - \frac{U_C^2}{4} + \frac{U_C}{2}\).

For the bottom curved part we set \(X_C = W - Y\) and \(X_D = 0\) and we get \(U_C = W - Y + 2\sqrt{Y}\) and \(U_D = \sqrt{Y}\). So, the bottom part is described as \(U_C = W - U_D^2 + 2U_D\).