Quasi-concave functions and concave functions.

- If f is concave, then it is quasi-concave, so you might start by checking for concavity.
- If f is a monotonic transformation of a concave function, it is quasi-concave. This also means that if a monotonic transformation of f is concave, then f is concave.
- Example: Check whether the $f(x, y) = xy + x^2y^2 + x^3y^3$ defined on $\Re^2_+$ is quasiconcave. Note that $f(x) = g(u(x, y))$ where $u(x, y) = xy$ and $g(z) = z + z^2 + z^3$. Since $g' > 0$, f is quasi-concave if and only if u is quasi-concave. But $u(x, y) = e^{v(x,y)}$ where $v(x, y) = \ln x + \ln y$. The function v is easily seen to be concave. So then

$$f(x) = g(u(x, y)) = g(e^{v(x,y)})$$

is a monotone increasing function of a concave function and hence is quasi-concave.

Necessary condition for quasi-concave function.

- Let f be a twice continuously differentiable function of n real variables. The bordered Hessian matrix of f looks like this.

$$H(x) = \begin{bmatrix} 0 & f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1(x) & f_{11}(x) & f_{12}(x) & \dots & f_{1n}(x) \\ f_2(x) & f_{21}(x) & f_{22}(x) & \dots & f_{2n}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n(x) & f_{n1}(x) & f_{n2}(x) & \dots & f_{nn}(x) \end{bmatrix}$$

- A necessary condition for f to be a quasi-concave function is that the even-numbered principle minors of the bordered Hessian be non-negative and the odd-numbered principle minors be non-positive.
- A sufficient condition for f to be quasi-concave is that the even-numbered principle minors of the bordered Hessian be strictly positive and the odd-numbered principle minors be strictly negative.

Supporting hyperplane theorem

- If X is a convex subset of $\Re^n$ and x_0 is a point in the boundary of X, then there exists a non-zero vector $p \in \Re^n$ such that $px \geq px_0$ for all $x \in X$.
- Suppose preferences are convex. Then $X = \succeq (x_0)$ is a convex set. If preferences are monotonic, then x_0 is on the boundary of X. Then according to the theorem, there is some p such that if $x \succeq x_0$, then $px \geq px_0$.

Separating hyperplane theorem

- If X and Y are disjoint, non-empty convex subsets of $\Re^n$, then there exists a non-zero vector $p \in \Re^n$ and a scalar b such that $px \geq b$ for all $x \in X$ and $py \geq b$ for all $x \in Y$.