Some Old Prelim Questions

1. Mr. Green consumes two goods, \(X\) and \(Y\). His utility function is

\[ U(x, y) = \ln \left( x + 2y - \frac{y^2}{2} \right) \]

where \(x\) is his consumption of \(X\) and \(y\) is his consumption of \(Y\). Let Good \(X\) be the numeraire with a price of 1 and let \(p\) be the price of Good \(Y\). Let \(w\) be Mr. Green’s income and assume that \(w > 1\).

A Ms. Blue’s preferences over bundles of \(X\) and \(Y\) are represented by the utility function

\[ U(x, y) = x + 2y - \frac{y^2}{2}. \]

How do Ms Blue's preference compare with Mr. Green’s? Explain.

B Define homothetic preferences. Does Mr. Green have homothetic preferences? If so, prove it. If not, show that he does not.

C Define convex preferences. Show that Ms. Blue has convex preferences. (In your proof feel free to use the fact that \(f(y) = 2y - y^2\) is a concave function.) Does Mr. Green have convex preferences?

D Find Mr. Green’s Walrasian demand functions \(x(p, w)\) and \(y(p, w)\) for the two goods \(X\) and \(Y\) when the price of \(X\) is 1, the price of \(Y\) is \(p\) and income is \(w\). Be careful to specify demand at price-income combinations that lead to a “corner solution” where Mr. Green buys no \(Y\). Note: We have made life easier for you by assuming that \(w > 1\). This ensures that there is no corner solution where Mr. Green buys no \(X\). (For extra credit if you have extra time, you could show that if \(w > 1\), then \(x(p, w) > 0\) for all \(p > 0\).)

E Find Mr. Green’s indirect utility function \(v(p, w)\). (Be careful to specify the indirect utility function for price-income combinations that lead to “corner solutions” as well those that lead to interior solutions.)

F What does Roy’s law tell us about the relation between the indirect utility function and the Walrasian demand function for \(Y\). Verify by direct calculation that Roy’s law is satisfied by Mr. Green’s demand function for \(Y\).

G Find Mr. Green’s expenditure function \(e(p, u)\) where the price of \(X\) is 1 and the price of \(Y\) is \(p\).

H Find Mr. Green’s Hicksian demand function \(h_y(p, u)\) for commodity \(Y\) where the price of \(X\) is 1 and the price of \(Y\) is \(p\).
2. Consider a consumer who consumes three goods and has utility function

\[ U(x_1, x_2, x_3) = (x_1 + b_1)^\alpha (x_2 + b_2)^\beta (x_3 + b_3)^\gamma \]

where \( b_i \geq 0 \) for \( i = 1, \ldots, 3 \) and where \( \alpha > 0, \beta > 0, \) and \( \gamma > 0. \)

A) Why can you assume that \( \alpha + \beta + \gamma = 1 \) without loss of generality?

B) Does this consumer have homothetic preferences? Explain

C) Does this consumer have additively separable preferences? Explain

D) Find this consumer’s demand function. Don’t forget to account for corner solutions if they exist.

E) Find this consumer’s indirect utility function and verify that Roy’s identity holds in this case.

F) Find this consumer’s Hicksian demand function and the substitution matrix.

G) Consider an economy with two types of people. While their utility functions are all of the form described above, they have different values of the \( b_i \)’s. Type 1 has \( b_1 = b_2 = b_3 = 10 \) and Type 2 has \( b_1 = 10, \) and \( b_2 = b_3 = 0. \) Persons of Type 1 have incomes \( M_1 \) and persons of type 2 have incomes \( M_2. \) If there are 100 people of each type, write an expression for the aggregate demand for good 1 as a function of the prices and incomes.

H) Suppose that initially prices and income are such that everybody buys positive amounts of all good. Income is redistributed from type 2’s to type 1’s and after the income redistribution everyone is still buying positive amounts of all goods. What can you say about the change in aggregate demand for good 1. Explain your answer.

I) Suppose that the prices are \( p_1 = p_2 = p_3 = 1. \) Find incomes \( M_1 \) and \( M_2 \) such that a transfer of income from type 1’s to type 2’s changes the aggregate demand for good 1. Explain your answer.
3. Harry consumes just one commodity and he will live for $T$ periods. His current preferences over consumption streams are represented by a utility function of the form

$$U(x_1, \ldots, x_T) = \sum_{t=1}^{T} \beta_t u(x_t)$$

where $x_t$ is the amount of the commodity that he will consume in year $t$ and where the function $u(\cdot)$ is strictly concave and twice continuously differentiable. Harry knows that his income stream will be $(w_1, \ldots, w_T)$ where $w_t$ is the income that he will receive in period $t$. Harry is able to borrow or lend at the constant interest $r$. At time 1, Harry is able to commit himself to any time path of consumption that satisfies his budget constraint. His budget constraint is that the present value of his lifetime consumption must not exceed the present value of his lifetime income stream.

Part 1) Suppose that for some $\alpha$ where $0 < \alpha < 1$, and for all $t = 1, \ldots, T$, $\beta_t = \alpha^{t-1}$. At what interest rate will Harry choose to consume the same amount of goods in every period of his life? Explain why your answer is correct. Does this interest rate depend on the time path of his income stream? At this interest rate, what can you say about the way in which his borrowing and saving behavior depend on the time path of his income stream.

Part 2) Suppose that $\beta_2 = \beta_1 = 1$ and that for $t = 3, \ldots, T$, $\beta_t = \alpha^{t-2}$. Suppose that at time 1, Harry can commit himself to a time path of future consumption. Qualitatively, how does his time path of consumption depend on the interest rate? For example, at what if any interest rates $r > 0$ is his consumption first increasing, then constant, at what interest rates is his consumption always increasing, at what interest rates is his consumption first increasing, then decreasing, etc, etc.

Part 3) Suppose that $T = 3$ and Harry’s utility function is

$$U(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{x_2} + \alpha \sqrt{x_3}.$$ 

Harry earns income $W > 0$ in period 1, while $w_t = 0$ for $t > 1$. Suppose also that

$$\frac{1}{1+r} = \alpha.$$ 

If Harry can commit himself to a time path of future consumption at time 1, solve for his choice of $x_1$, $x_2$, and $x_3$ as a function of the parameters $W$ and $r$.

Part 4) Suppose that Harry can save money in period 1 but he must leave the choice of allocation between periods 2 and 3 to his future self. Harry is aware of this and knows that in Period 2 his utility function for consumption periods 2 and 3 will be

$$U(x_2, x_3) = \sqrt{x_2} + \sqrt{x_3}.$$ 

He also knows that the interest rate will continue to satisfy the equation

$$\frac{1}{1+r} = \alpha.$$ 

3
If Harry consumes $x_1$ in period 1, what consumptions will his period 2 self choose for periods 2 and 3? Write down an expression for Harry’s utility as a function of $x_1$, taking into account the fact that he knows that his period 2 self will determine the division of income between his period 2 self and his period 3 self. Find the optimal choice of $x_1$ for Harry. Is this the same as the amount of $x_1$ that he would choose in Part 3 above?
4. Oskar consumes two goods $X$ and $Y$. He is an expected utility maximizer with von Neumann-Morgenstern utility function

$$u(x, y) = \sqrt{x - \frac{1}{y}}$$

where $x$ is his consumption of $X$ and $y$ is his consumption of $Y$.

A Suppose that Oskar can only buy “sure thing” consumptions, that is, the amounts he consumes are determined with certainty. Suppose that the price of a unit of $X$ is 1 and the price of a unit of $Y$ is $p$. If Oskar’s income is $M$, write expressions for Oskar’s demands for $X$ and for $Y$ as functions of $p$ and $M$. (Be sure to specify when there is a corner solution and when there is an interior solution).

B If the only consumption bundles available are sure thing bundles, and if the price of $X$ is 1, the price of $Y$ is $p$, and income is $M$, write Oskar’s indirect utility function as a function of $p$ and $M$.

C Suppose that the price of good $X$ is certain to be 1 and the price of good $Y$ is certain to be 4, but that income is a random variable which takes on value $M_1$ if event 1 happens and $M_2$ if event 2 happens. Suppose that event 1 happens with probability $\pi$ and event 2 with probability $1 - \pi$. Write a utility function that represents Oskar’s preferences among alternative lotteries which differ in the values of $M_1$, $M_2$, and $\pi$.

D As before, the price of good $X$ is certain to be 1 and the price of good $Y$ is certain to be 4, but income is a random variable which takes on value $M_1$ if event 1 happens and $M_2$ if event 2 happens. Event 1 happens with probability $\pi$ and event 2 with probability $1 - \pi$. Write a “certainty equivalent utility function” for Oskar, showing the amount of money to be had with certainty that is just indifferent to a lottery that awards $M_1$ in event 1 and $M_2$ in event 2.