Relative and Absolute Risk Aversion

Question 1.
A) Define the Arrow-Pratt measure of absolute risk aversion.
Answer: Where $u$ is the von Neumann-Morgenstern utility function,

$$R_A(y) = \frac{-u''(y)}{u'(y)}.$$ 

B) Consider the following von Neumann Morgenstern utility function

$$u(x) = -\frac{1}{\alpha} e^{-\alpha x}.$$ 

For what values of $\alpha$ is a consumer with this utility function risk-averse? Does this consumer display increasing, decreasing, or constant absolute risk aversion? Explain.
Answer: This consumer is risk averse if and only if $\alpha > 0$. For this function, $R_A(y) = \alpha$. Since $\alpha$ does not change with $y$, this consumer has constant absolute risk aversion.

C) Consider the following von Neumann Morgenstern utility function

$$u(x) = \frac{1}{\alpha} x^\alpha.$$ 

For what values of $\alpha$ is a consumer with this utility function risk-averse? Does this consumer display increasing, decreasing, or constant absolute risk aversion? Does this consumer display increasing, decreasing, or constant relative risk aversion?
Answer: In this case, $u''(x) = (\alpha - 1)x^{\alpha - 2}$. We see that $u''(x) < 0$ so long as $\alpha < 1$, so he is risk averse if and only if $\alpha < 1$. Taking derivatives and simplifying, we find that $R_A(x) = \frac{1-\alpha}{x}$ Differentiating with respect to $x$, we see that he has decreasing absolute risk aversion if $\alpha < 1$.

This consumer’s index of relative risk aversion is

$$R_R(x) = xR_A(x) = (1-\alpha)$$

Therefore he has constant relative risk aversion.

D) Ulrich and Virgil have twice-differentiable von Neumann Morgenstern utility functions $u(x)$ and $v(x)$. Virgil’s utility function is given by $v(x) = f(u(x))$ where $f(\cdot)$ is a strictly increasing and strictly concave function. Prove that Virgil is strictly more risk averse than Ulrich by the Arrow-Pratt measure of risk aversion.
Answer: Let $v(x) = f(u(x))$. Take derivatives of both sides to find that

$$v'(x) = f'(u(x))u'(x).$$
Do it again to find that
\[ v''(x) = f''(u(x))u'(x) + f'(u(x))u''(x) \]
. Then
\[ \frac{v''(x)}{v'(x)} = \frac{f''(x)}{f'(x)} + \frac{u''(x)}{u'(x)}. \]

Therefore when Ulrich’s index of relative risk aversion is \( R_{Au}(x) \), Virgil’s index of absolute risk aversion is
\[ R_{Av}(x) = -\frac{v''(x)}{v'(x)} = -\frac{f''(x)}{f'(x)} \cdot \frac{u''(x)}{u'(x)} = -\frac{f''(x)}{f'(x)} + R_{Au}(x), \]

Since \( f \) is strictly increasing and strictly concave, it must be that \( -\frac{f''(x)}{f'(x)} > 0 \) and therefore Virgil has a higher index of relative risk aversion than Ulrich.

**Exchange Equilibrium with Alice and Bob**

**Question 3.**

Consider a pure exchange economy with two consumers, Alice and Bob and two goods, \( X \) and \( Y \). Alice is endowed with \( \omega_x \) units of \( X \) and no good \( Y \). Bob is endowed with \( \omega_y \) units of \( Y \) and no good \( X \). Alice’s utility function is
\[ U_A(x_A, y_A) = x_A + C \ln y_A \]
where \( C > 0 \) and \( x_A \) and \( y_A \) are her consumptions of \( X \) and \( Y \). Bob’s utility function is
\[ U_B(x_B, y_B) = x_B y_B \]
where \( x_B \) and \( y_B \) are his consumptions of \( X \) and \( Y \). Let \( X \) be the numeraire and let \( p \) be the price of good \( Y \).

**A)** Solve for Alice’s demand for \( X \) as a function of \( p \).

**Answer:** Alice’s marginal rate of substitution between \( Y \) and \( X \) equals the price ratio when \( y = \frac{C}{p} \). From her budget constraint we see that this implies that her demand for \( X \) is
\[ D_A(p) = \omega_x - C, \]
for all \( \omega_x > C \).

**B)** Solve for Bob’s demand for \( X \) as a function of \( p \).

**Answer:** Bob’s income is \( p \omega_y \) and his demand for \( X \) is therefore
\[ D_B(p) = \frac{p \omega_y}{2}. \]
C) Write an equation that says that demand for good \( X \) equals supply of good \( X \) and solve for the price \( \bar{p} \) at which demand equals supply for good \( X \). At this price, what do we know about the excess demand for good \( Y \)?

**Answer:** Supply of \( X \) equals demand for \( X \) when

\[
\omega_x = D_A(\bar{p}) + D_B(\bar{p}) = \omega_x - C + \frac{\bar{p}\omega_y}{2}.
\]

This is true if and only if

\[
\bar{p} = \frac{2C}{\omega_y}.
\]

D) How much good \( X \) does Alice consume in competitive equilibrium? How much good \( Y \) does Alice consume in equilibrium?

**Answer:** Alice consumes \( D_A(\bar{p}) = \omega_x - C \) units of \( X \) and she consumes

\[
\frac{C}{\bar{p}} = \frac{\omega_y}{2}
\]

units of \( Y \).

**Intertemporal Decisions of Gordon Grasshopper**

**Question 5.** Gordon Grasshopper consumes two goods \( X \) and \( Y \) and he will survive for three periods. Let \( x_t \) and \( y_t \) be his consumptions in period \( t \) of good \( X \) and \( Y \) respectively. Gordon’s preferences over time paths of consumption is given by the intertemporal utility function

\[
U(x_1, x_2, x_3, y_1, y_2, y_3) = (x_1y_1)^{k/2} + \frac{1}{2}(x_2y_2)^{k/2} + \frac{1}{4}(x_3y_3)^{k/2}
\]

where \( 0 < k < 1 \). Gordon has an endowment of $W$ in the first period and will receive no income in the other periods. He can, however, save money for future consumption. The interest rate is zero.

A) Find the indirect utility function in the case where there are just two goods and one time period for a person who has the utility function

\[
u(x, y) = (xy)^{k/2} \text{ where } k > 0.
\]

**Answer:** The indirect utility function is

\[
v(p_x, p_y, m) = u\left(\frac{m}{2p_x}, \frac{m}{2p_y}\right) = 2^{-k} (p_xp_y)^{-k/2} m^k.
\]

B) Suppose that the prices of goods \( x \) and \( y \) are \( p_x(1) = p_y(1) = 1 \) in period 1, \( p_x(2) = 1/4 \) and \( p_y(2) = 4 \) in period 2, and \( p_x(3) = 1/4, p_y(3) = 4 \) in period 3.
Use your answer from part A to help you to write an expression for Gordon’s intertemporal utility as a function of the amounts of money $m_1$, $m_2$, and $m_3$ that he spends in each period.

**Answer:** Gordon’s utility over alternative expenditure flows, $m_1, m_2, m_3$ are represented by

$$v(1,1,m_1) + \frac{1}{2} v\left(\frac{1}{2}, 2\right) + \frac{1}{4} v\left(\frac{1}{4}, 4\right) = 2^{-k}\left(m_1^k + \frac{1}{2}m_2^k + \frac{1}{4}m_3^k\right).$$

**C)** Suppose that $k = 2/3$. Use your answer to Part B and Gordon’s budget constraint to solve for the amount of money that Gordon will spend in each period.

**Answer:** Gordon’s marginal rate of substitution between money spent in any two periods is 1. Thus we must have

$$\frac{1}{2} \frac{m_2^{k-1}}{m_1^{k-1}} = \frac{1}{4} \frac{m_3^{k-1}}{m_1^{k-1}} = 1.$$  

Rearranging terms and setting $k = 1/3$, we find that

$$m_2 = \frac{1}{8} m_1$$

and

$$m_3 = \frac{1}{64} m_1.$$  

We also must have

$$m_1 + m_2 + m_3 = W.$$  

Therefore it must be that $m_1 = \frac{64}{73} W$, $m_2 = \frac{8}{73} W$, and $m_3 = \frac{1}{73} W$.

**D)** How would Gordon allocate his expenditures if $k = 1$?

**Answer:** In this case we have the full grasshopper solution, $m_1 = W$, $m_2 = m_3 = 0$. 
