Gorman Polar Form

When we do equilibrium analysis, whether with supply and demand curves or with more advanced techniques, we are interested in aggregate demand and aggregate supply for goods. The aggregate demand for any good j is the sum of the individual demands for that good. In general, aggregate demand is the sum of the individual Marshallian demand functions.

$$D_j(p, m_1, \ldots, m_n) = \sum_{i=1}^{n} x_j^i(p, m_i).$$

If there are m commodities and n consumers, we see that D_j is a function of $n + m$ variables. In general equilibrium theory, we consider models in which the incomes m_i are determined by the price vector p and thus demands are determined by the prices alone. Applied economists sometimes like to shortcut this process by assuming that demand for any good depends only on prices and the sum of incomes. That is, if $m = \sum_i m_i$ is aggregate income, they assume that the aggregate demand function for good j be

$$X_j(p, m) = \sum_i x_j^i(p, m_i).$$

This assumption is often euphemized by calling it the "representative consumer model". Whatever you call it, this is a very strong assumption, that can lead to misleading conclusions. Here we examine circumstances under which the representative consumer model is appropriate.

Indirect utility is said to be of the Gorman polar form if for all consumers i, $v(p, m_i) = A^i(p) + B(p)m_i$ for some functions $A^i(p)$ and $B(p)$ (over an appropriately restricted domain). It turns out aggregate demand is independent of income distribution if and only if indirect utility can be represented by functions of the Gorman polar form.

Something to notice. To say that indirect utility can be represented by some *utility functions of the Gorman polar form is not to say that* every *utility function that represents indirect utiliy must be of the Gorman polar form. For example, suppose that $v(p, m_i)$ is of the Gorman polar form. The function $v^*(p, m_i) = v(p, m_i)^3$ is a strictly increasing transformation of v and hence also represents indirect utility. But v^* is* not *of the Gorman polar form.*

Applying Roy's law, we see that over the range of prices and incomes such that i buys positive amounts of all goods, i's Marshallian demand function demand for good j is given by

$$x_j^i(p, m_i) = \frac{-A_j^i(p)}{B(p)} - \frac{B_j(p)}{B(p)} m_i$$

where

$$A_j^i = \frac{\partial A^i(p)}{\partial p_j}$$

and

$$B_j(p) = \frac{\partial B(p)}{\partial p_j}.$$

Then

$$\begin{aligned} X_j(p,m) &= -\sum_i \left(\frac{A_j^i(p)}{B(p)} + \frac{B_j(p)}{B(p)} m_i\right) && (1)\\ &= \sum_i \frac{A_j^i(p)}{B(p)} + \sum_i \left(\frac{B_j(p)}{B(p)}\right) m_i && (2)\\ &= \sum_i \frac{A_j^i(p)}{B(p)} + \frac{B_j(p)}{B(p)} \sum_i m_i && (3)\end{aligned}$$

Note that these demand functions all have linear income expansion paths. Since the functions $B_j(p)$ are the same for all i, the slopes of all consumers' income expansion paths are the same. But the intercepts are not necessarily the same, since the functions $A(p)$ may be different.

This turns out to be an if and only if result. Proving the converse is a little more difficult and we won't do it here.

Example: A Quasi-linear case

Let $u^i(x_1, x_2) = x_1 + 2c_i\sqrt{x_2}$. Then

$$x_2^i(p, m_i) = \left(\frac{c_i p_1}{p_2}\right)^2$$

and

$$x_1^i(p, m_i) = \frac{m_i}{p_1} - \frac{c_i^2 p_1}{p_2}.$$

Then

$$\begin{aligned} v^i(p,m) &= \frac{m_i}{p_1} - \frac{c_i^2 p_1}{p_2} + 2c_i\sqrt{x_2(p,m_i)} && (4)\\ &= \frac{m}{p_1} + \frac{c_i^2 p_1}{p_2} && (5)\end{aligned}$$

which is of the Gorman polar form with $B(p) = 1/p_1$ and $A^i(p) = \frac{c_i^2 p_1}{p_2}$.

Aggregate demand for good 2 is

$$\sum_i \left(\frac{c_i p_1}{p_2}\right)^2$$

and aggregate demand for good 1 is

$$\frac{\sum_i m_i}{p_1} - \sum_i \frac{c_i^2 p_1}{p_2}.$$

A Stone-Geary example

Let

$$u^i(x_1, x_2) = (x_1 + c_i)^{1/2} x_2^{1/2}.$$

We find that

$$v^i(p, m) = \frac{(m + p_1 c_i)}{2\sqrt{p_1 p_2}}$$

Identical homothetic utilities

Suppose $u^i(x_1, x_2) = u(x_1, x_2)$ for all i and u is homothetic. Then preferences can be represented by a function u^* that is homogeneous of degree 1. Demand is $x(p, m) = mx(p)$ for some function $x(p)$. Indirect utility can be represented by $v(p, m) = u^*(x(p, m)) = mu^*(x(p)) = B(p)m$ where $B(p) = u^*(x(p))$.