Envelope theory for constrained optimization

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Envelope theory shows us how to deal with the interplay of direct and indirect effects of parameters in a constrained maximization (or minimization) problem:

Consider the following problem:

Choose x to maximize (or minimize) $f(x,a)$ subject to the constraint that $g(x,a) \leq b$. where x is an n vector, a is an m vector, and b is a scalar. Think of a, b as a vector of "parameters" for the problem, x as a "choice vector" and f as a payoff function. Assume that f and g are twice continuously differentiable functions and that a unique maximum exists $x(a,b) = (x_1(a,b) \ldots x_n(a,b)$ exists for all a and b in in some open set.

To avoid complications, let us assume that f and g are both strictly increasing functions of x.[1]

Define $v(a,b) = f(x(a,b))$. The $v(a,b)$ is the maximum payoff attained with the parameters a, b. We know from earlier considerations that at a maximum, it must be that for some scalar, λ, and for all $i = 1, \ldots n$,

$$\frac{\partial f(x(a,b),a)}{\partial x_i} = \lambda \frac{\partial g(x(a,b),a)}{\partial x_i} \tag{1}$$

and also that for all a and b in the domain,

$$g(x(a,b)) = b \tag{2}$$

Since Equation 2 holds for all b, the derivative of the left side equals that of the right side, so we must have

$$\sum_i \frac{\partial g(x(a,b),a)}{\partial x_i} \frac{\partial x_i(a,b)}{\partial b} = 1 \tag{3}$$

Similarly, since Equation 2 holds for all a, it must be that for every $j = 1, \ldots m$, the derivative of the left side with respect to a_j is equal to that of the right side. This tells us that

$$\sum_i \frac{\partial g(x(a,b),a)}{\partial x_i} \frac{\partial x_i(a,b)}{\partial a_j} + \frac{\partial g(x(a,b),a)}{\partial a_j} = 0 \tag{4}$$

Now we know that

$$v(a,b) = f(x(a,b),a). \tag{5}$$

Lets see what we can learn about the derivatives of v. First we have

$$\frac{\partial v(a,b)}{\partial b} = \frac{\partial f(x(a,b),a)}{\partial b} \tag{6}$$

$$= \sum_i \frac{\partial f(x(a,b),a)}{\partial x_i} \frac{\partial x_i(a,b)}{\partial b} \tag{7}$$

[1]This assumption can be greatly relaxed.

$$
\begin{aligned}
&= \lambda \sum_i \frac{\partial g\,(x(a,b),a)}{\partial x_i} \frac{\partial x_i(a,b)}{\partial b} && (8)\\
&= \lambda && (9)
\end{aligned}
$$

where the step from Equation 7 to Equation 8 follows from Equation 1 and the step from 8 to 9 follows from Equation 3.

Next let us take derivatives with respect to a_k. We have

$$
\begin{aligned}
\frac{\partial v(a,b)}{\partial a_k} &= \frac{df\,(x(a,b),a)}{da_k} && (10)\\
&= \sum_i \frac{\partial f\,(x(a,b),a)}{\partial x_i} \frac{\partial x_i(a,b)}{\partial a_k} + \frac{\partial f\,(x(a,b),a)}{\partial a_k} && (11)\\
&= \lambda \sum_i \frac{\partial g\,(x(a,b),a)}{\partial a_k} \frac{\partial x_i(a,b)}{\partial a_k} + \frac{\partial f\,(x(a,b),a)}{\partial a_k} && (12)\\
&= -\lambda \frac{\partial g\,(x(a,b),a)}{\partial a_k} + \frac{\partial f\,(x(a,b),a)}{\partial a_k} && (13)\\
&= -\frac{\partial v(a,b)}{\partial b} \frac{\partial g\,(x(a,b),a)}{\partial a_k} + \frac{\partial f\,(x(a,b),a)}{\partial a_k} && (14)
\end{aligned}
$$

So we have

$$
\frac{\partial v(a,b)}{\partial a_k} = -\frac{\partial v(a,b)}{\partial b} \frac{\partial g\,(x(a,b),a)}{\partial a_k} + \frac{\partial f\,(x(a,b),a)}{\partial a_k}. \qquad (15)
$$

Why is this interesting? Notice that Equation 15 involves only the derivatives of the function v and the "direct" effects of the parameter a_k on the payoff function and on the constraint function. It doesn't involve any terms that relate to the indirect effects of changes in the x_i's. If the functional form of the direct effects is fairly simple, we can make useful inferences.

Let us consider two familiar examples.

Example 1: The standard consumer budget problem. In this case we interpret x be a vector of n commodities, a as a vector of n prices and b as income. Then in our earlier notation, $f(x,a) = u(x)$ and $g(x,a) = \sum a_i x_i$ and $x(a,b)$ is the demand vector that maximizes $u(x)$ subject to $px \le b$. In this case, we see that

$$
\frac{\partial f(x,a)}{\partial a_k} = 0
$$

for all k. (Consumers don't care directly about prices, but only about how they affect their budgets.) We also see that

$$
\frac{\partial g(x(a,b),a)}{\partial a_k} = x_k(a,b).
$$

Therefore the equation 15 tells us that

$$
\frac{\partial v(a,b)}{\partial a_k} = -\frac{\partial v(a,b)}{\partial b} x_k(a,b)
$$

which if let $p = a$ and $m = b$ implies that

$$x_k(p,m) = -\frac{\partial v(p,m)}{\partial p_k} \div \frac{\partial v(p,m)}{\partial m}.$$

and so we are able to find the Marshallian demand functions just by taking partial derivatives of the indirect utility function.

Example 2: Cost minimization problem Again, let x be a vector of n commodities, a as a vector of n prices and b as income. This time, let $f(x,a) = \sum_i a_i x_i$, let b represent a specified utility level u and let $g(x,a) = u(x)$. Then $x(a,b)$ is the solution to the constrained minimization problem Minimize $f(x,a)$ subject $g(x,a) = b$. The value of the solution $v(a,b)$ is this time the "expenditure function"' $e(p,u)$ where $e(p,u)$ is the cost of achieving utility u at prices p.

This time the direct effects are

$$\frac{\partial f(x(a,b),a)}{\partial a_k} = x_k(a,b)$$

and

$$\frac{\partial g(x(a,b),a)}{\partial a_k} = 0.$$

So when we apply Equation 15, we have

$$\begin{aligned} \frac{\partial e(p,u)}{\partial p_k} &= \frac{\partial e(p,u)}{\partial u} \times 0 + x_k(p,u) && (16) \\ &= x_k(p,u) && (17) \end{aligned}$$

The solution $x(p,u)$ to this problem is known as the "Hicksian demand function" and the function $x_k(p,u)$ is the Hicksian demand for good k.