# Lecture Notes on Elasticity of Substitution

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Today's featured guest is "the elasticity of substitution."

## Elasticity of a function of a single variable

Before we meet this guest, let us spend a bit of time with a slightly simpler notion, the elasticity of a a function of a single variable. Where $f$ is a differentiable real-valued function of a single variable, we define the elasticity of $f(x)$ with respect to $x$ (at the point $x$) to be

$$\eta(x) = \frac{xf'(x)}{f(x)}. \tag{1}$$

Another way of writing the same expression 1 is

$$\eta(x) = \frac{x\frac{df(x)}{dx}}{f(x)} = \frac{\frac{df(x)}{f(x)}}{\frac{dx}{x}}. \tag{2}$$

From Expression 2, we see that the elasticity of of $f(x)$ with respect to $x$ is the ratio of the percent change in $f(x)$ to the corresponding percent change in $x$.

Measuring the responsiveness of a dependent variable to an independent variable in percentage terms rather than simply as the derivative of the function has the attractive feature that this measure is invariant to the units in which the independent and the dependent variable are measured. For example, economists typically express responsiveness of demand for a good to its price by an elasticity.[1] In this case, the percentage change in quantity is the

[1] Some economists find it tiresome to talk about negative elasticities and choose to define the price-elasticity as the absolute value of the percentage responsiveness of quantity to price.

same whether quantity is measured in tons or in ounces and the percentage change in price is the same whether price is measured in dollars, Euros, or farthings. Thus the price elasticity is a "unit-free" measure. For similar reasons, engineers measure the stretchability of a material by an "elasticity" of the length of the material with respect to the force exerted on it.

The elasticity of the function $f$ at a point of $x$ can also be thought of as the slope of a graph that plots $\ln x$ on the horizontal axis and $\ln f(x)$ on the vertical axis. That is, suppose that we make the change of variables $u = \ln x$ and $v = \ln y$ and we rewrite the equation $y = f(x)$ as $e^v = f(e^u)$. Taking derivatives of both sides of this equation with respect to $u$ and applying the chain rule, we have

$$e^v \frac{dv}{du} = e^u f'(e^u) \tag{3}$$

and hence

$$\frac{dv}{du} = \frac{e^u f'(e^u)}{e^v} = \frac{x f'(x)}{f(x)} = \eta(x), \tag{4}$$

where the second equality in Expression 4 is true because $e^u = x$ and $e^v = f(x)$. Thus $\frac{dv}{du}$ is the derivative of $\ln f(x)$ with respect to $\ln x$. We sometimes express this by saying that

$$\eta(x) = \frac{d \ln f(x)}{d \ln x}. \tag{5}$$

It is interesting to consider the special case where the elasticity of $f(x)$ with respect to $x$ is a constant, $\eta$ that does not dependent on $x$. In this case, integrating both sides of Equation 5, we have

$$\ln f(x) = \eta \ln x + a \tag{6}$$

for some constant $a$. Exponentiating both sides of Equation 6, we have

$$f(x) = cx^\eta \tag{7}$$

where $c = e^a$. Thus we see that $f$ has constant elasticity $\eta$ if and only if $f$ is a "power function" of the form 7.

In general, the elasticity of $f$ with respect to $x$ depends on the value of $x$. For example if $f(x) = a - bx$, then $\eta(x) = \frac{-bx}{a-bx}$. In this case, as $x$ ranges from 0 to $a/b$, $\eta(x)$ ranges from 0 to $-\infty$.

## Elasticity of inverse functions

Another useful fact about elasticities is the following. Suppose that the function $f$ is either strictly increasing or strictly decreasing. Then there is a well defined inverse function $\phi$, defined so that such that $\phi(y) = x$ if and only if $f(x) = y$. It turns out that if $\eta(x)$ is the elasticity of $f(x)$ with respect to $x$, then $1/\eta(x)$ is the elasticity of $\phi(y)$ with respect to $y$.

*Proof.* Note that since the function $\phi$ is the inverse of $f$, we must have $\phi(f(x)) = x$. Using the chain rule to differentiate both sides of this equation with respect to $x$, we see that if $y = f(x)$, then $\phi'(y)f'(x) = 1$ and hence $\phi'(y) = 1/f'(x)$. Therefore when $y = f(x)$, the elasticity of $\phi(y)$ with respect to $y$ is

$$\frac{y\phi'(y)}{\phi(y)} = \frac{f(x)}{\phi(y)f'(x)}.$$

But since $y = f(x)$, it must be that $\phi(y) = x$, and so we have

$$\frac{y\phi'(y)}{\phi(y)} = \frac{f(x)}{xf'(x)} = \frac{1}{\eta(x)}.$$

□

## Application to monopolist's revenue function

One of the most common applications of the notion of elasticity of demand is to monopoly theory, where a monopolist is selling a good and the quantity of the good that is demanded is a function $D(p)$ of the monopolist's price $p$. The monopolist's revenue is $R(p) = pD(p)$. Does the monopolist's revenue increase of decrease if he increases his price and how is this related to the price elasticity? We note that $R(p)$ is increasing (decreasing) in price if and only if $\ln R(p)$ increases (decreases) as the log of price increases. But $\ln R(p) = \ln p + \ln D(p)$. Then

$$\frac{d \ln R(p)}{d \ln p} = 1 + \frac{d \ln D(p)}{d \ln p} = 1 + \eta(p)$$

where $\eta(p)$ is the price elasticity of demand. So revenue is an increasing function of $p$ if $\eta(p) > -1$ and a decreasing function of $p$ if $\eta < -1$. In the former case we say demand is inelastic and in the latter case we say demand is elastic.

## Elasticity of substitution

Now we introduce today's main event– the elasticity of substitution for a function of two variables. The elasticity of substitution is most often discussed in the context of production functions, but is also very useful for describing utility functions. A firm uses two inputs (aka factors of production) to produce a single output. Total output $y$ is given by a concave, twice differentiable function $y = f(x_1, x_2)$. Let $f_i(x_1, x_2)$ denote the partial derivative (marginal product) of $f$ with respect to $x_i$. While the elasticity of a function of a single variable measures the percentage response of a dependent variable to a percentage change in the independent variable, the elasticity of substitution between two factor inputs measures the percentage response of the relative marginal products of the two factors to a percentage change in the ratio of their quantities.

The elasticity of substitution between any two factors can be defined for any concave production function of several variables. But for our first crack at the story it is helpful to consider the case where there are just two inputs and the production function is homogeneous of some degree $k > 0$. We also assume that the production function is differentiable and strictly quasi-concave.

**Fact 1.** *If $f(x_1, x_2)$ is homogeneous of some degree $k$ and strictly quasi-concave, then the ratio of the marginal products of the two factors is determined by the ratio $x_1/x_2$ and $f_1(x_1, x_2)/f_2(x_1, x_2)$ is a decreasing function of $x_1/x_2$.*

*Proof.* If $f$ is homogeneous of degree k, then the partial derivatives of $f$ are homogeneous of degree $k - 1$.[2] Therefore

$$f_1(x_1, x_2) = x_2^{k-1} f_1\left(\frac{x_1}{x_2}, 1\right)$$

and

$$f_2(x_1, x_2) = x_2^{k-1} f_2\left(\frac{x_1}{x_2}, 1\right).$$

It follows that

$$\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{x_2^{k-1} f_1\left(\frac{x_1}{x_2}, 1\right)}{x_2^{k-1} f_2\left(\frac{x_1}{x_2}, 1\right)} \tag{8}$$

[2] To prove this, note that if $f$ is homogeneous of degree $k$, then $f(\lambda x) = \lambda^k f(x)$. Differentiate both sides of this equation with respect to $x_i$ and arrange terms to show that $f_i(\lambda x) = \lambda^{k-1} f_i(x)$.

$$= \frac{f_1\left(\frac{x_1}{x_2}, 1\right)}{f_2\left(\frac{x_1}{x_2}, 1\right)}. \tag{9}$$

Therefore the ratio of marginal products is determined by the ratio $x_1/x_2$. Let us define this ratio as

$$g\left(\frac{x_1}{x_2}\right) = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}.$$

Since strict quasi-concavity implies diminishing marginal rate of substitution, it must be that $g$ is a strictly decreasing function of $x_1/x_2$.

□

Since $g$ is strictly decreasing, it must be that the function $g$ has a well-defined inverse function. Let's call this inverse function $h$.

Let prices of the two inputs be given by the vector $p = (p_1, p_2)$. Suppose that the firm always chooses factors so as to minimize its costs, conditional on its output level. Then it must be that at prices $p$, the firm uses factors in the ratio $x_1/x_2$ such that

$$g(x_1/x_2) = \frac{f_1\left(\frac{x_1}{x_2}, 1\right)}{f_2\left(\frac{x_1}{x_2}, 1\right)} = \frac{p_1}{p_2}$$

or equivalently such that

$$\frac{x_1}{x_2} = g^{-1}\left(\frac{p_1}{p_2}\right) = h\left(\frac{p_1}{p_2}\right).$$

The elasticity of substitution is just the negative of the elasticity of the function $h$ with respect to its argument $p_1/p_2$. That is,

$$\sigma(\frac{p_1}{p_2}) = -\frac{\frac{p_1}{p_2} h'\left(\frac{p_1}{p_2}\right)}{h\left(\frac{p_1}{p_2}\right)} = -\frac{d \ln h\left(\frac{p_1}{p_2}\right)}{d \ln\left(\frac{p_1}{p_2}\right)}. \tag{10}$$

As we remarked in our earlier discussion, the elasticity of an inverse function is just the inverse of the elasticity of a function. The function $g$ defined in Equation  is the inverse of the function $h$ defined in Equation  and so where

$$\frac{x_1}{x_2} = h\left(\frac{p_1}{p_2}\right)$$

it must be that the elasticity $\sigma(x_1/x_2)$ of the function $g$ satisfies the equations

$$\frac{1}{\sigma(p_1/p_2)} = -\frac{d \ln \frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}}{d \ln \frac{x_1}{x_2}} \tag{11}$$

## Constant Elasticity of Substitution

A very interesting special class of production functions is those for which the elasticity of substitution is a constant $\sigma$. These have come to be known as CES utility functions. This class of functions was first explored in a famous paper published in 1961 by Arrow, Chenery, Minhas, and Solow [1].[3] These authors prove that a production function with $n$ inputs has constant elasticity of substitution $\sigma$ between every pair of inputs if and only if the production function is either of the functional form

$$f(x_1, \ldots, x_n) = A \left( \sum_{i=1}^{n} \lambda_i x_i^{\rho} \right)^{k/\rho} \tag{12}$$

or else of the Cobb-Douglas form

$$A \prod_{i=1}^{n} x_i^{\lambda_i} \tag{13}$$

where $A > 0$, $k > 0$, where $\lambda_i \geq 0$ for all $i$, $\sum_i \lambda_i = 1$ and where $\rho$ is a constant, possibly negative.

This function is readily seen to be homogeneous of degree $k$. It is also easy to check that the form in equation 12 has constant elasticity of substitution $\sigma = 1/(1-\rho)$ between any two variables and that in equation 13 has constant elasticity $\sigma = 1$. The proof of the converse result that a CES utility function must be of one of these two forms is not very difficult, but we will not show it here.

[3]This paper is a notable example of the interaction of theoretical advances and empirical data for understanding economic phenomena. Though in retrospect, its results seem pretty straightforward, they were a revelation at the time. Two of the authors, Arrow and Solow, went on to win Nobel prizes.

## Cost functions for CES production functions

In this discussion, we assume there are only two factors. These results extend in a pretty obvious way to the case of a CES function with $n$ factors. Since $f$ has a constant elasticity of substitution, it must be that:

$$\ln\left(\frac{x_2}{x_1}\right) = \sigma \ln\left(\frac{f_1(x)}{f_2(x)}\right) + \mu. \tag{14}$$

where $\sigma$ is the constant elasticity of substitution and $\mu$ is a constant.

Define cost functions $c(w, y)$. Constant returns to scale implies $c(p, y) = c(w, 1)y$ where $c(w, 1)$ is the cost of producing one unit. By Shepherd's Lemma the conditional factor demand for good $i$ is given $x_i(w, y) = c_i(w, 1)y$ where $c_i(w, 1)$ is the partial derivative of $c(w, 1)$ with respect to $w_i$. Cost minimization requires that the ratio of marginal products is equal to the ratio of prices, so

$$\frac{f_1\left(x(w, y)\right)}{f_2\left(x(w, y)\right)} = \frac{w_1}{w_2}.$$

According to Shepherd's lemma,

$$\frac{c_2(w, 1)}{c_1(w, 1)} = \frac{x_2}{x_1}.$$

From Equation 14 it follows that

$$\ln\left(\frac{c_2(w, 1)}{c_1(w, 1)}\right) = \sigma \ln\left(\frac{w_1}{w_2}\right) + \mu \tag{15}$$

Rearranging terms of equation 15, we find that

$$\ln\left(\frac{w_2}{w_1}\right) = \frac{1}{\sigma} \ln\left(\frac{c_1(w, 1)}{c_2(w, 1)}\right) + \frac{\mu}{\sigma} \tag{16}$$

This proves the following result:

**Fact 2.** *If the production function $f(x_1, x_2)$ has constant elasticity of substitution $\sigma$ between factors 1 and 2, then the cost function $c(w_1, w_2)$ must have constant elasticity of substitution, $1/\sigma$ between the prices of factors 1 and 2.*

**Finding a CES cost function**

Suppose that

$$f(x_1, x_2) = (\lambda_1 x_1^\rho + \lambda_2 x_2^\rho)^{1/\rho}$$

is a CES production function with $\lambda_1 + \lambda_2 = 1$. Then

$$\ln\left(\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}\right) = (\rho - 1)\ln\left(\frac{x_1}{x_2}\right) + \ln\left(\frac{\lambda_1}{\lambda_2}\right). \tag{17}$$

Rearranging terms (and noting that $\ln(x_2/x_1) = -\ln(x_1/x_2)$), we have

$$\begin{aligned} \ln\left(\frac{x_2}{x_1}\right) &= \frac{1}{1-\rho}\ln\left(\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}\right) - \frac{1}{1-\rho}\ln\left(\frac{\lambda_1}{\lambda_2}\right) \\ &= \sigma\ln\left(\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}\right) - \sigma\ln\left(\frac{\lambda_1}{\lambda_2}\right), \end{aligned} \tag{18}$$

where $\sigma = 1/(1-\rho)$ is the elasticity of substitution of $f$.

Let $c(w_1, w_2)y$ be the corresponding cost function. We have shown that $c(w_1, w_2)$ is a constant elasticity of substitution function with elasticity of substitution $1/\sigma$. Therefore the function $c$ must be of the form

$$c(w_1, w_2) = (a_1 w_1^r + a_2 w_2^r)^{1/r}. \tag{19}$$

We have seen that a function of this form has elasticity of substitution $1/(1-r)$. We have shown that the elasticity of substitution of $c$ must also equal $1/\sigma$ where $\sigma$ is the elasticity of substitution of the original production function. Therefore

$$\frac{1}{\sigma} = \frac{1}{1-r}. \tag{20}$$

We can solve for $r$ in terms of the parameter $\rho$ of the original production function. From Equation 20 it follows that $\sigma = 1 - r$ and hence

$$r = 1 - \sigma = 1 - \frac{1}{1-\rho} = \frac{\rho}{\rho - 1}.$$

We still have a bit more work to do. How are the constants $a_1$ and $a_2$ in the cost function related to the coefficients in the production function? One way to find this out is as follows. At the wage rates, $w_1 = \lambda_1$ and $w_2 = \lambda_2$, the cheapest way to produce 1 unit is to set

$$x_1(\lambda_1, \lambda_2) = x_2(\lambda_1, \lambda_2) = 1$$

and the cost of producing one units is $c(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 = 1$. (To see this, note that when one unit of each factor is used, the ratio of marginal products is equal to $\lambda_1/\lambda_2$ and total output is equal to $\lambda_1 + \lambda_2 = 1$.)

Thus, we know that

$$c(\lambda_1, \lambda_2) = (a_1\lambda_1^r + a_2\lambda_2^r)^{1/r} = 1. \tag{21}$$

Calculating derivatives, we have

$$\frac{c_1(\lambda_1, \lambda_2)}{c_2(\lambda_1, \lambda_2)} = \frac{a_1}{a_2}\left(\frac{\lambda_1}{\lambda_2}\right)^{r-1} \tag{22}$$

We also have, from Shepherd's lemma

$$\frac{c_1(\lambda_1, \lambda_2)}{c_2(\lambda_1, \lambda_2)} = \frac{x_1(\lambda_1, \lambda_2)}{x_2(\lambda_1, \lambda_2)} = 1 \tag{23}$$

From Equations 22 and 23 it follows that

$$\frac{a_1}{a_2} = \left(\frac{\lambda_1}{\lambda_2}\right)^{1-r} \tag{24}$$

From Equations 21 and 24 it then follows that $a_1 = \lambda_1^{1-r}$ and $a_2 = \lambda_2^{1-r}$.

From Equation 19 it follows that if the production function is

$$f(x_1, x_2) = (\lambda_1 x_1^\rho + \lambda_2 x^\rho)^{1/\rho}$$

then the cost of producing one unit of output is given by the function

$$c(w_1, w_2, 1) = \left(\lambda_1^{1-r}w_1^r + \lambda_2^{1-r}w_2^r\right)^{1/r} \tag{25}$$

where

$$r = \frac{\rho}{\rho - 1}.$$

We previously found that $r = 1 - \sigma$ where $\sigma$ is the elasticity of substitution of the production function. Therefore the following result follows from Equation 25 and the fact that with constant returns to scale, $c(w_1, w_2, y) = c(w_1, w_2, 1)y$.

**Fact 3.** *The production function*

$$f(x_1, x_2) = (\lambda_1 x_1^\rho + \lambda_2 x_2^\rho)^{1/\rho}$$

*with* $\lambda_1 + \lambda_2 = 1$ *has constant elasticity of substitution* $\sigma = 1/(1-\rho)$. *The corresponding cost function has elasticity of substitution* $1/\sigma = (1-\rho)$ *and is given by*

$$c(w_1, w_2, y) = \left(\lambda_1^\sigma w_1^{1-\sigma} + \lambda_2^\sigma w_2^{1-\sigma}\right)^{1/(1-\sigma)} y.$$

We can also work backwards. If we know that if there is a CES cost function with parameter $r$, then it corresponds to a CES production function with parameter $\rho = r/(1-r)$.

So for example if $\rho = -1$, $r = 1/2$ and if $r = 1/2$, $\rho = -1$.

Remark: We have solved for the cost function if

$$f(x_1, x_2) = (\lambda_1 x_1^\rho + \lambda_2^\rho)^{1/\rho}$$

where $r \leq 1$. It is not hard now to find the cost function for the more general CES function

$$f(x_1, x_2) = A\left(\lambda_1 x_1^\rho + \lambda_2^\rho\right)^{k/\rho}$$

where $A > 0$ and $k > 0$. Hint: If $f$ is homogeneous of degree 1, how is the cost function for the production function $g$ such that $g(x) = Af(x)^k$ related to the cost function for $f$?

# Generalized means, CES functions, and limiting cases

These notes are based on the presentation by Hardy, Littlewood and Polya in their classic book *Inequalities* [2]. Consider a collection of numbers $x = \{x_1, \ldots, x_n\}$ such that $x_i \geq 0$ for all $i$. We define the *mean of order* $r$ of these numbers as

$$M_r(x) = \left(\frac{1}{n}\sum_i x_i^r\right)^{1/r} \tag{26}$$

If $r < 0$, then the above definition of $M_r(x_1, \ldots, x_n)$ does not apply when $x_j = 0$ for some $j$, since $0^r$ is not defined for $r < 0$. Therefore for $r < 0$, we define $M_r(x_1, \ldots, x_n)$ as in Equation 30 if $x_i > 0$ for all $i$ and we define $M_r(x_1, \ldots, x_n) = 0$ if $x_j = 0$ for some $j$.

The special case where $r = 1$ is the familiar *arithmetic mean* of the numbers $x_1, \ldots, x_n$. The case where $r = -1$ is known as the *harmonic mean.*

So far, we have not defined $M_0(x_1, \ldots, x_n)$. We can demonstrate the following useful fact.

**Fact 4.**

$$lim_{r\to 0}\left(\sum a_i x_i^r\right)^{1/r} = \prod_{i=1}^{n} x_i^{a_i}.$$

*Proof.* To prove this, note that

$$\ln\left(\sum a_i x_i^r\right)^{1/r} = \frac{1}{r}\ln\left(\sum a_i x_i^r\right) \tag{27}$$

Applying L'Hospital's rule to the expression on the right, we have

$$\begin{aligned} \lim_{r\to 0}\frac{1}{r}\ln\left(\sum a_i x_i^r\right) &= \ln\left(\lim_{r\to 0}\frac{\frac{d}{dr}\left(\sum a_i x_i^r\right)}{\frac{d}{dr}r}\right) \\ &= \lim_{r\to 0}\sum a_i x_i^r \ln x_i \\ &= \sum a_i \ln x_i. \end{aligned} \tag{28}$$

Therefore

$$\lim_{r\to 0}\frac{1}{r}\ln\left(\sum a_i x_i^r\right) = \sum a_i \ln x_i.$$

Since the exponential function is continuous, it must be that

$$\begin{aligned} \lim_{r\to 0}\left(\sum a_i x_i^r\right)^{1/r} &= e^{\lim_{r\to 0}\frac{1}{r}\ln\left(\sum a_i x_i^r\right)} \\ &= e^{\sum a_i \ln x_i} \\ &= \prod_{i=1}^{n} x_i^{a_i} \end{aligned} \tag{29}$$

□

The special case where $r = 0$ for all $i$ is known as the *geometric mean.*

Two other interesting limiting cases of means of order $r$ are the limits as $r$ approaches $\infty$ and $-\infty$. We have the following result:

**Fact 5.**

$$\lim_{r\to\infty} M_r(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$$

*and*

$$\lim_{r\to -\infty} M_r(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$$

Here are some more useful facts about generalized means.

**Fact 6.** *For any real number $r$, $M_r(x_1, \ldots, x_n)$ is homogeneous of degree one and is a non-decreasing function of each of its arguments. Moreover,*

$$\min\{x_1, \ldots, x_n\} \leq M_r(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}.$$

**Fact 7.** *If $r < s$, then unless all of the $x_i$ are equal,*

$$M_r(x) < M_s(x).$$

Immediate consequences of Fact 7 are that for any collection of numbers that are not all equal, the geometric mean is less than the arithmetic mean and the harmonic mean is less than the geometric mean.

We have demonstrated the following fact in previous discussions.

**Fact 8.** *The function $M_r(x)$ is a concave function if $r \leq 1$ and a convex function if $r \geq 1$.*

This result is sometimes known as *Minkowski's inequality*.

A simple generalization of the means of order $r$ is the *weighted mean of order $r$*. Where $a = a_1, \ldots, a_n)$ such that $a_i \geq 0$ for all $i$ and $\sum_i a_i = 1$, we define the function $M_r(a, x)$ of $x = \{x_1, \ldots, x_n\}$ so that

$$M_r(a, x) = \left(\sum a_i x_i^r\right)^{1/r}. \tag{30}$$

Thus the complete family of CES functions used by economists can be regarded as weighted means of some order $r$.

The properties that we have found for ordinary means extend in a straightforward way to weighted means. Hardy, Littlewood and Polya point out that not only are ordinary means special cases of weighted means, but weighted means with rational weighs can be constructed as ordinary means by "replacing every number (in $x$) by an appropriate set of equal numbers."

# References

[1] Kenneth J. Arrow, Hollis B. Chenery, Bagicha S. Minhas, and Robert M. Solow. Capital-labor substitution and economic efficiency. *Review of Economics and Statistics*, 43(3):225–250, August 1961.

[2] G. Hardy, J.E. Littlewood, and Pólya G. *Inequalities*. Cambridge University Press, Cambridge, U.K., 1934.