Proving that a Cobb-Douglas function is concave if the sum of exponents is no bigger than 1

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If you tried this problem in your homework, you learned from painful experience that the Hessian conditions for concavity of the Cobb-Douglas function

$$F(x_1, \ldots x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}$$

from $\Re^n_+$ to $\Re$ are cumbersome to work with, when $n \geq 3$. Maybe you were thinking "There must be an easier way." Well, there is. Indeed there is more than one other way to skin this cat, but the way that I will show you here is instructive and in the process you will pick up a couple of useful tools.

It turns out that the function F is a concave function if $\alpha_i \geq 0$ for all i and $\sum_{i=1}^{n} \alpha_i \leq 1$.

I propose the following road to a proof.

We first note the following:

Lemma 1. *The function defined by*

$$F(x_1, \ldots x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}$$

is homogeneous of degree $\sum_{i=1}^{n} \alpha_i$.

You should be able to supply the proof of this lemma.

We next note that F is quasi-concave. To show this, we make use of the fact that any monotone increasing transformation of a concave function is quasi-concave.

Lemma 2. *A function F is quasi-concave if $h(x) = g(F(x))$ is a concave function for some strictly increasing function g from $\Re$ to $\Re$.*

You should be able to prove this. First show that if h is concave, then h must also be quasi-concave. Then show that a monotone increasing function of a quasi-concave function must be quasi-concave.

Suppose we let $g(x) = \ln x$ and $h(x) = g(F(x) = \ln F(x) = \sum_{i=1}^{n} \alpha_i \ln x_i$. You should be able to show that $\sum_{i=1}^{n} \alpha_i \ln x_i$ is a concave function. (For this one, the Hessian second-order condition is really easy. The off-diagonals of the Hessian matrix are zeros.) Therefore $F(x)$ is a monotone transformation of a concave function and hence must be ~~concave~~.

Theorem 1 is an important result to know about. For our application, it tells us that the Cobb-Douglas function F is a concave function if $\sum_i \alpha_i = 1$.

Theorem 1. *Let f be a real-valued function defined on $\Re^n_+$ (the nonnegative orthant in Euclidean n-space) and suppose that f is quasi-concave and homogeneous of degree 1. Then f is a concave function.*

Proof. Suppose that f is quasi-concave and homogeneous of degree 1. Consider any two points x and x' in $\Re^n_+$. Since f is homogeneous of degree 1, it must be that

$$f\left(\frac{x}{f(x)}\right) = \frac{1}{f(x)} f(x) = 1$$

and

$$f\left(\frac{x'}{f(x')}\right) = \frac{1}{f(x')} f(x') = 1.$$

Since f is quasi-concave, it must be that for all θ such that $0 < \theta < 1$,

$$f\left(\theta \frac{x}{f(x)} + (1-\theta)\frac{x'}{f(x')}\right) \geq \min\left\{f\left(\frac{x}{f(x)}\right), f\left(\frac{x'}{f(x')}\right)\right\} = 1. \quad (1)$$

For any t such that $0 < t < 1$, let

$$\theta = \frac{tf(x)}{tf(x) + (1-t)f(x')}.$$

Then

$$1 - \theta = \frac{(1-t)f(x')}{tf(x) + (1-t)f(x')}.$$

Substituting these expressions for θ and $1-\theta$ into the inequality 1, we have the inequality

$$f\left(\frac{tx + (1-t)x'}{tf(x) + (1-t)f(x')}\right) \geq 1 \quad (2)$$

Since f is homogeneous of degree 1, it follows

$$f\left(\frac{tx + (1-t)x'}{tf(x) + (1-t)f(x')}\right) = \frac{1}{tf(x) + (1-t)f(x')} f(tx + (1-t)x')$$

and therefore the inequality 2 implies that

$$f(tx + (1-t)x') \geq tf(x) + (1-t)f(x').$$

which means that f is a concave function.

□

What if $0 < \sum \alpha_i < 1$? Theorem 2 gives you the tool you need to handle this case.

Theorem 2. *Let f be a real-valued function defined on a convex set X in R^n and let g be an increasing concave function from the $\Re$ to $\Re$. Let $h(x) = f(g(x))$. Then $h(x)$ is a concave function.*

Proof. If f is a concave function, then for any t such that $0 < t < 1$ and for any x and x' in X,

$$f(tx + (1-t)x') \geq tf(x) + (1-t)f(x'). \tag{3}$$

Since g is an increasing function, it follows from inequality 3 that

$$g\left(f(tx + (1-t)x')\right) \geq g\left(tf(x) + (1-t)f(x')\right). \tag{4}$$

Since g is a concave function, it must be that

$$g\left(tf(x) + (1-t)f(x')\right) \geq tg\left(f(x)\right) + (1-t)g\left(f(x')\right). \tag{5}$$

Combining the inequalities 3 and 5, we have

$$g\left(f(tx + (1-t)x')\right) \geq tg\left(f(x)\right) + (1-t)g\left(f(x')\right) \tag{6}$$

Recalling the definition of h, we see that the inequality 6 can be written as

$$h\left(tx + (1-t)x'\right) \geq th(x) + (1-t)h(x'). \tag{7}$$

But the inequality 7 is the condition for h to be a concave function.

□

How does Theorem 2 help? Let $k = \sum \alpha_i$. We can verify that the Cobb-Douglas function F must be homogeneous of degree k. Define $H(x) = F(x)^{1/k}$. We note that H is homogeneous of degree 1 and quasi-concave. Therefore H is homogeneous of degree 1. From Theorem 1 we know that H is concave. Now $F(x) = g(H(x))$ where $g(y) = y^k$. The second-derivative test shows us that g is a concave function. So it follows from Theorem 2 that F is a concave function.

There we are.

This proof was kind of a long road, but I think a very instructive one. Everything that you learned along the way is likely to come in handy some day.

A final remark

I leave it to you to show that if $\sum_i \alpha_i > 1$, then F is neither concave nor convex.