Questions and Answers from Econ 210A Final: Fall 2008

I have gone to some trouble to explain the answers to all of these questions, because I think that there is much to be learned by working through them. Please let me know if you find mistakes or inadequate explanations.
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Question 1) Firm A has production function

$$f(x_1, x_2) = \left(x_1^{1/2} + x_2^{1/2}\right)^{2k}$$

where $k > 0$.

a) For what values of k is f a quasi-concave function? For what values of k is f a concave function? Explain your answers.

Firm B has production function

$$g(x_1, x_2) = \left(x_1^2 + x_2^2\right)^{k/2}$$

where $k > 0$.

b) For what values of k is g a quasi-concave function? For what values of k is g a concave function? Explain your answers.

Answers to Question 1

Answer to 1a: The easiest way to check for quasi-concavity of f is to remember that a function is quasi-concave if and only if every monotonic increasing transformation of that function is quasi-concave. In particular, the function

$$f(x_1, x_2) = \left(x_1^{1/2} + x_2^{1/2}\right)^{2k}$$

is a monotonic increasing transformation of the function

$$g(x_1, x_2) = x_1^{1/2} + x_2^{1/2},$$

But it is easy to verify that g is quasi-concave. You could do this by checking the bordered Hessian conditions, which are now pretty simple because there are no off-diagonal terms in the regular Hessian part of the bordered Hessian. You could make the problem even easier by checking that g is a concave function and then using the fact that a concave function must be quasi-concave. Or you could alternatively just note that the isoquants for g have diminishing marginal rate of substitution.

When is the function f concave? We could check this by writing out the Hessian matrix and making sure that it is negative semi-definite. But there is an easier way. We see that the function f is homogeneous of degree k. We know that it is quasi-concave, and our textbook tells us that if a function is quasi-concave and homogeneous of degree 1 it is concave. Suppose that $k < 1$. Define the function $h(x) = f(x)^{1/k}$. We know that $h(x)$ is homogeneous of degree one and quasi-concave, so it is concave. But now $f(x) = h(x)^k$ where $k < 1$. This means that f is a concave function of a concave function and hence must be concave.

What if f is homogeneous of degree $k > 1$? Then it will not be a concave function. There are increasing returns to scale. It is easy to check that f is not a concave function. One could prove this by showing a single example of two points that violate the condition for concavity For example let $x = (0, 0)$ and $x' = (1, 0)$. Then $f(x) = 0$, $f(x') = 1$. Now for any $t \in (0, 1)$, $tx' + (1 - t)x = (t, 0)$. We have $(1 - t)f(x) + tf(x') = t$. But $f((1 - t)x + tx') = f(t, 0) = t^k$. If $k > 1$ and $0 < t < 1$, we see that $t^k < t$. Therefore $f((1 - t)x + tx') < (1 - t)f(x) + tf(x') = t$, which means that f is not a concave function.

Answer to 1b:

This one should be really easy if you think about it. The isoquants for this production function have the equation $x_1^2 + x_2^2 =$ Constant. What do they look like? Can this function be quasi-concave? No! This should be obvious once you look at an isoquant. To be a little more formal about this: Much as we remarked in Part 1a) the function f will be quasi-concave if and only if $g(x_1, x_2) = x_1^2 + x_2^2$ is quasi-concave. This function is not quasi-concave. You can verify that it is not quasi-concave by checking the bordered Hessian condition, or more easily by showing an example that violates quasi-concavity. Like $x = (0, 1)$ $x' = (1, 0)$. Then $f(x) = f(x') = 1$, but $f(x/2 + x'/2) = 1/2 < (1/2)f(x) + (1/2)f(x')$.

Since we know that a concave function must be quasi-concave, the function f is not concave for any $k > 0$.

Question 2) Firm A, described in Problem 1, is a price-taker in the factor market and must pay w_1 per unit of factor 1 and w_2 per unit of factor 2 that it uses.

a) Find its conditional input demand functions $x_1(w_1, w_2, 1)$ and $x_2(w_1, w_2, 1)$ for producing one unit of output.

b) Find its conditional input demand functions $x_1(w_1, w_2, y)$ and $x_2(w_1, w_2, y)$ for producing y units of output in the case where $k = 1$.

c) Find its conditional input demand functions $x_1(w_1, w_2, y)$ and $x_2(w_1, w_2, y)$ for producing y units of output for arbitrary k.

Answers to Question 2

Answer to 2a

$$x_1(w_1, w_2, 1) = \left(\frac{1}{1 + \frac{w_1}{w_2}}\right)^2 = \left(\frac{w_2}{w_1 + w_2}\right)^2$$

$$x_2(w_1, w_2, 1) = \left(\frac{w_1}{w_1 + w_2}\right)^2$$

Answer to 2b When $k = 1$,

$$x_1(w_1, w_2, y) = \left(\frac{w_2}{w_1 + w_2}\right)^2 y$$

$$x_2(w_1, w_2, y) = \left(\frac{w_1}{w_1 + w_2}\right)^2 y$$

Answer to 2c For general $k > 0$,

$$x_1(w_1, w_2, y) = \left(\frac{w_2}{w_1 + w_2}\right)^2 y^{1/k}$$

$$x_2(w_1, w_2, y) = \left(\frac{w_1}{w_1 + w_2}\right)^2 y^{1/k}$$

Question 3) Firm B, described in Problem 1, is a price-taker in the factor market and must pay w_1 per unit of factor 1 and w_2 per unit of factor 2 that it uses.

a) Find its conditional input demand functions $x_1(w_1, w_2, 1)$ and $x_2(w_1, w_2, 1)$ for producing one unit of output.

b) Find its conditional input demand functions $x_1(w_1, w_2, y)$ and $x_2(w_1, w_2, y)$ for producing y units of output in the case where $k = 1$.

Answers to Question 3

If $w_1 < w_2$, firm will use only factor 1, so $x_2(w_1, w_2, y) = 0$ and if $k = 1$, $x_1(w_1, w_2, y) = y$. If $w_2 < w_1$, $x_1(w_1, w_2, y) = 0$ and if $k = 1$, $x_2(w_1, w_2, y) = y$.

Question 4)

a) Find the cost function $c(w_1, w_2, y)$ of Firm A for the case where $k = 1$.

b) Verify that Shephard's lemma is satisfied in the case of Firm A.

c) Find the cost function $c(w_1, w_2, y)$ of Firm B for the case where $k = 1$.

Answers to Question 4

Answer to 4a: You can find the cost function for producing one unit by the following steps:

$$\begin{aligned}
c(w_1, w_2, 1) &= w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y) \\
&= \frac{w_1 w_2^2}{(w_1 + w_2)^2} + \frac{w_2 w_1^2}{(w_1 + w_2)^2} \\
&= \frac{w_1 w_2 (w_1 + w_2)}{(w_1 + w_2)^2} \\
&= \frac{w_1 w_2}{w_1 + w_2} \\
&= \left(w_1^{-1} + w_2^{-1}\right)^{-1}
\end{aligned}$$

Alternatively, you can make use of the fact that if a production function has constant elasticity of substitution $\sigma = 1/(1-\rho)$, then the corresponding cost function is a ces function of the form $(w_1^r + w_2^r)^{1/r}$ where $1/\ sigma = 1/(1-r)$ and hence where $r = 1 - \sigma = -\rho/(1-\rho)$. In our case, where $\rho = 1/2$, $r = -1$.

Then for $k = 1$,

$$c(w_1, w_2, y) = yc(w_1, w_2, 1) = y\left(w_1^{-1} + w_2^{-1}\right)^{-1}.$$

Answer to 4b: To verify Shephard's lemma, we calculate

$$\begin{aligned}\frac{\partial c(w_1, w_2, y)}{\partial w_1} &= yw_1^{-2}\left(w_1^{-1} + w_2^{-1}\right)^{-2} \\ &= yw_1^{-2}\left(\frac{w_1 w_2}{w_1 + w_2}\right)^2 \\ &= y\left(\frac{w_2}{w_1 + w_2}\right)^2 \\ &= x_1(w_1, w_2, y)\end{aligned}$$

A symmetrical argument would work for the demand for factor 2.

Answer to 4c: Remember that firm B uses only the cheaper of the two inputs and when it does so, its output is exactly equal to the quantity of its input. Thus $c(w_1, w_2, y) = y \min\{w_1, w_2\}$.

Question 5) a) There are two goods in the economy, X and Y. All consumers have identical utility functions of the form

$$U(x, y) = x + 2y^{1/2},$$

where x is the quantity of X consumed and y the quantity of Y consumed. There are n consumers and consumer i has income m_i. What is the demand function of consumer i for good Y? For what prices does consumer i demand a positive amount of good Y?

b) What is the aggregate demand for good Y at prices in the range for which all consumers demand positive amounts of Y? What is the price elasticity of demand for good Y when all consumers are choosing positive consumptions of Y?

c) Assume that all consumers demand positive amounts of good Y at all relevant prices. If Y is produced by a monopolist who has zero fixed costs and constant marginal costs of c, what price will the monopolist charge? How many units will he sell?

d) Suppose that the Y market is served by two Cournot duopolists, both of which have constant marginal costs of c. In Cournot equilibrium, what price will each charge and how many units will each sell.

Answers to Question 5

Answer to 5a: Let's make good X the numeraire so that its price is 1 and let p be the price of Y. The first order condition for an interior max is

$$p = y^{-1/2}.$$

If this is the case, then $y = p^{-2}$. What about corner solutions? There will never be a corner solution where $y = 0$, since the marginal utility of y approaches infinity as y approaches 0. But there will be a corner solution with $x = 0$ if $pp(-2 = p^{-1} > m_i$ or equivalently if $p < 1/m$. If $p > 1/m$, there will be an interior solution.

Answer to 5b: With n consumers, aggregate demand will be np^{-2}. The price elasticity of demand is therefore -2.

Answer to 5c: A monopolist will price so that marginal revenue equals marginal cost. For a monopolist with constant elasticity of demand η, marginal revenue is equal price times $(1 + 1/\eta)$. In this case, the monopolist chooses price p such that $p(1 - 1/2) = c$, which means simply that $p = 2c$. Then his total sales will be $n(2c)^{-2}$.

Answer to 5d: We showed in class that Cournot oligopolists choose quantities so that

$$p(1 + \frac{1}{n}\frac{1}{\eta}) = c$$

where c is marginal cost, n is the number of oligopolists in the industry and η is the elasticity of demand for the final good. In this case, we have $n = 2$ and η is the elasticity of demand. In our case we have $n = 2$ and $\eta = -2$. This implies that $p(3/4) = c$ and hence $p = (4/3)c$.

6) There are three goods in the economy, X_0, X_1, and X_2. Goods X_1 and X_2 are used by consumers as inputs into a "household production function" which produces a final consumer good Z. The household production function is

$$z = f(x_1, x_2) = \left(x_1^{1/2} + x_2^{1/2}\right)^2$$

where z is the amount of output of good Z and x_1 and x_2 are household inputs of goods 1 and 2. There are n consumers and each consumer has a utility function of the form

$$U(x_0, z) = x_0 + 2z^{1/2}.$$

The price of Good X_0 is 1 dollar per unit. Consumers cannot buy Z but can buy goods X_1 and X_2 at prices p_1 and p_2 respectively and use these goods as inputs to produce Z in their households. Consumer i has income m_i dollars. Let x_0 be the number of units of good X_0 consumed and let y be the number of dollars spent on good Z. Then consumer i's budget constraint is $x_0 + y = m_i$.

a) Suppose that the prices of goods X_1 and X_2 are p_1 and p_2. If consumer i spends y dollars on these goods, how much good X_1 should he buy? how much good X_2?

b) Suppose we define $v(p_1, p_2, y)$ to be the maximum amount of Good Z that a consumer can produce if the prices of X_1 and X_2 are p_1 and p_2 and if she spends a total of y dollars on goods X_1 and X_2. Write down an explicit function for $v(p_1, p_2, y)$.

c) Given your solution to Part b), show how a consumer's utility maximization problem can be broken into two stages. In Stage A one decides how much money to spend on the inputs for good Z. In stage B one decides how to allocate this expenditures between goods X_1 and X_2. Explain which stage you have to "solve first" and how you would proceed.

d) Carry out the procedure described in Part c) for the specific production function in this problem. (Hint: In part b) you found $v(p_1, p_2, y)$ which was the total amount of Z that you could get if you spent y on the inputs for Z.) Assuming that the best choice is an interior solution, how much money y will a consumer with income m_i choose to spend on the inputs for Good z? How much does he buy of each of the input goods? (Your answers will be functions of p_1 and p_2.)

Answers to Question 6

Answer to 6a: To maximize output for a given expenditure, the firm must set the marginal rate of substitution between the two goods equal to the price ratio. This happens when

$$\left(\frac{x_1}{x_2}\right)^{-1/2} = \frac{p_1}{p_2},$$

which implies that

$$x_2 = \left(\frac{p_1}{p_2}\right)^2 x_1.$$

The budget constraint requires that

$$p_1 x_1 + p_2 x_2 = y.$$

It follows that

$$\begin{aligned} x_1(p_1, p_2, y) &= \frac{y}{p_1 + p_2 \frac{p_1^2}{p_2^2}} \\ &= \frac{p_2 y}{p_1 p_2 + p_1^2} \\ &= \frac{p_2}{p_1}\left(\frac{y}{p_1 + p_2}\right). \end{aligned} \tag{1}$$

By a symmetric argument, we have

$$x_2(p_1, p_2, y) = \frac{p_1}{p_2}\left(\frac{y}{p_1 + p_2}\right).$$

Answer to 6b: Now substitute these expressions into the function F to find the "indirect utility" $v(p_1, p_2, y)$. The main thing to see is that $v(p_1, p_2, y)$ is a function that is linear in y and homogeneous of degree minus one in prices. But it is interesting to see that the function simplifies rather nicely.

$$\begin{aligned}
v(p_1, p_2, y) &= y\left(\left(\frac{p_2}{p_1(p_1 + p_2)}\right)^{1/2} + \left(\frac{p_1}{p_2(p_1 + p_2)}\right)^{1/2}\right)^2 \\
&= \frac{y}{p_1 + p_2}\left(\left(\frac{p_2}{p_1}\right)^{1/2} + \left(\frac{p_1}{p_2}\right)^{1/2}\right)^2 \\
&= \frac{y}{p_1 + p_2}\left(\left(\frac{p_2^2}{p_1 p_2}\right)^{1/2} + \left(\frac{p_1^2}{p_1 p_2}\right)^{1/2}\right)^2 \\
&= \frac{y}{(p_1 + p_2)p_1 p_2}(p_1 + p_2)^2 \\
&= y\frac{p_1 + p_2}{p_1 p_2} \\
&= y\left(\frac{1}{p_1} + \frac{1}{p_2}\right).
\end{aligned}$$

A brief digression

Just to show you that there is more than one way to skin a cat, we note that instead of finding $v(p_1, p_2, y)$ directly, we could have used duality theory and the expenditure function. Earlier we calculated the expenditure function corresponding to the production function F. We found that the cost of producing z units when factor prices were p_1 and p_2 was $e(p_1, p_2, z) = z\left(\frac{1}{p_1} + \frac{1}{p_2}\right)^{-1}$. But according to duality theory, $e(p_1, p_2, v(p_1, p_2, y)) = y$. But that means that

$$v(p_1, p_2, y)\left(\frac{1}{p_1} + \frac{1}{p_2}\right)^{-1} = y,$$

which in turn implies that

$$v(p_1, p_2, y) = y\left(\frac{1}{p_1} + \frac{1}{p_2}\right).$$

Answer to 6c: To solve the consumer's problem in two stages, we note that for any amount of money spent on the inputs to home production, you will spend that money so as to maximize the amount produced. You solve the production problem first to find that amount that you will be able to produce is $v(p_1, p_2, y)$. Therefore if you spend y on home production inputs, you will be able to purchase $m_i - y$ units of good 0 and your utility will be $m_i - y + 2(v(p_1, p_2, y))^{1/2}$. You would choose y to maximize this function.

Part d: For our particular problem this amounts to choosing y to maximize

$$m_i - y + 2\left(\frac{1}{p_1} + \frac{1}{p_2}\right)^{1/2} y^{1/2}$$

The calculus first order condition for this maximization is

$$1 = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)^{1/2} y^{-1/2}$$

which is equivalent to

$$y = \frac{1}{p_1} + \frac{1}{p_2} = \frac{p_1 + p_2}{p_1 p_2}.$$

This is the amount of money that a consumer chooses to spend on producing the household good. To find the amount that he spends on goods X_1 and X_2, we look at the conditional factor demands. The amount of X_1 that will be demanded at prices p_1 and p_2 is

$$\begin{aligned} x_1(p_1, p_2, y) &= y\left(\frac{p_2}{p_1}\right)\frac{1}{p_1 + p_2} \\ &= \left(\frac{p_1 + p_2}{p_1 p_2}\right)\left(\frac{p_2}{p_1}\right)\frac{1}{p_1 + p_2} \\ &= \frac{1}{p_1^2} \end{aligned}$$

A similar line of reasoning shows that when the amount of X_2 that will be demanded at these prices is

$$\frac{1}{p_2^2}.$$

A Remark for the curious: It is surprising that in this example, the demand for good X_1 depends only on p_1 and not on p_2 and vice versa. This is not true in general, but depends on the particular choice of parameters that I made for this problem.

Question 7) (extra credit–attempt only if you have done pretty well with Question 6.) Suppose that one monopolist controls the supply of X_1 and another monopolist controls the supply of X_2. Suppose that both of these goods are produced at constant marginal cost c. Suppose that each assumes that the other's price is unaffected by his own choice of price. Set up the problem and say as much as you can about the solution.

Answer to Question 7

In the answer to Question 6, we found that if the sellers of X_1 and X_2 set prices p_1 and p_2, then the quantity of X_i that will be purchased by each demander is $1/p_i^2$. We have assumed that there are constant marginal costs of c. If the X_1 sets his price at p_1, he will make a profit of $p_1 - c$ on each unit sold. Therefore his profits will be

$$\frac{p_1 - c}{p_1^2}.$$

We see that this function must have an internal maximum since profits are negative for $p_1 < c$ and approach zero from above when p_1 approaches infinity. Therefore there will be a profit maximum at a point where the derivative of profits with respect to p_1 is zero.

Simple calculation shows that the derivative of profits with respect to p_1 is equal to zero when

$$p^2 = 2pc.$$

There are two solutions to this equation $p = 0$ and $p = 2c$. The profit maximizing price can't be $p = 0$, so it must be $p = 2c$. In this case profits of the X_1 monopolist are $c/4c^2 = 1/4c$.

In equilibrium, the X_2 monopolist solves a similar problem and sets $p_2 = 2c$. One can then solve for the values of all of the other variables in the system. We would have for each consumer, $x_1 = x_2 = 1/4c^2$ and $z = f(x_1, x_2) = c$.

Other questions one might address: What would the prices and quantities be if a single monopolist controlled both goods X_1 and X_2? What if goods 1 and 2 were controlled by separate monopolies but that they assumed that each others' quantities rather than prices were invariant to their own decisions? What would the prices and quantities be if X_1 and X_2 were supplied competitively?