A DEFINITION OF SUBJECTIVE PROBABILITY[1]

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1. Introduction. It is widely recognized that the word "probability" has two very different main senses. In its original meaning, which is still the popular meaning, the word is roughly synonymous with plausibility. It has reference to reasonableness of belief or expectation. If "logic" is interpreted in a broad sense, then this kind of probability belongs to logic. In its other meaning, which is that usually attributed to it by statisticians, the word has reference to a type of physical phenomena, known as random or chance phenomena. If "physics" is interpreted in a broad sense, then this kind of probability belongs to physics. Physical probabilities can be determined empirically by noting the proportion of successes in some trials. (The determination is inexact and unsure, like all other physical determinations.)

In order to distinguish these two main senses, physical probabilities will be referred to as "chances," whereas "probability" unqualified will refer to logical probability.

Within the two main categories of logical probability (probability proper) and physical probability (chances), especially in the former, various lesser differences of meaning can be distinguished. In this paper we are concerned with the personal or subjective concept of probability, as considered by Ramsey [13] and Savage [14]. Probabilities and utilities are defined in terms of a person's preferences, in so far as these preferences satisfy certain consistency assumptions. The definition is constructive; that is, the probabilities and utilities can be calculated from observed preferences.

Some persons, especially those with scientific training, are acquainted with the mathematical theory of chances and consider it to be an adequate theory for some kinds of physical phenomena—the uncertain outcomes of the spin of a roulette wheel, the toss of a coin, the roll of a die, a random-number generator. They believe that equipment can be found whose output conforms well with the theory of chances, with stochastic independence between successive observations, and *with stated values* for the chances of the simple outcomes. It suffices for this purpose that they believe that some one such piece of equipment exists, for example, a fair coin, since any system of chances can be realized by multiple use of such equipment. (The relation of the theory of chances with chance phenomena has been well illustrated by Kerrich [6]. See also Neyman [11].)

For such a person, his utilities can be defined in terms of chances, as shown by von Neumann and Morgenstern [10]. The purpose of this note is to define

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the person's probabilities in terms of chances, by an extension of the von Neumann-Morgenstern theory. The addition of only two plausible assumptions to those of utility theory permits a simple and natural definition of probabilities having the appropriate properties.

The idea of defining subjective probability in some such way has been "in the air" for a decade or so; in Section 5 we discuss some of the relevant literature. We believe that our presentation of subjective probability is the simplest so far given, for anyone who accepts the physical theory of chances.

2. Lotteries. All of the following considerations are based on the preferences of a single individual, whom we call "you".

Let $\mathfrak{A}$ be a set of prizes. A *lottery* on $\mathfrak{A}$ is a device for deciding which prize in $\mathfrak{A}$ you will receive, on the basis of a single observation that records which one of a set of mutually exclusive and exhaustive uncertain events took place. It is possible that with each of these uncertain events there is associated a known chance; for example, this would be so if we were observing a single spin of a well-made roulette wheel. On the other hand, it is possible that chances cannot be associated with the uncertain events in question, or that the values of such chances are unknown; for example, this would be so if we were observing a horse race. To distinguish between the two kinds of lotteries, we call the first a "roulette lottery," and the second a "horse lottery." To fix ideas, we shall discuss only horse lotteries that are based on a single, particular horse race. Our object is to define the probabilities which you associate with each of the possible outcomes of this race.

In addition to the simple lotteries just described, we consider *compound* lotteries. These are constructed from simple lotteries by iteration; a lottery whose prizes are other lotteries is a compound lottery. A compound lottery may be compounded from roulette lotteries only, or from roulette lotteries and horse lotteries. Von Neumann-Morgenstern utilities of the basic prizes are constructed from preference comparisons between compound roulette lotteries. Subjective probabilities of the outcomes of the race will be constructed from preference comparisons between lotteries that are compounded from horse and roulette lotteries.

3. Assumptions. Utility theory for roulette lotteries based on $\mathfrak{A}$ becomes mathematically easier if one assumes that among the prizes there is a most desired prize, say A_1, and a least desired prize, say A_0, such that A_1 is definitely preferred to $A_0 (A_1 \succ A_0)$ and for any prize A the following two "preference or indifference" relations hold: $A_1 \succsim A$, $A \succsim A_0$. This assumption has been made by Luce and Raiffa [9]—in fact, they go further and assume that $\mathfrak{A}$ is finite, but that is unnecessary for their proofs—and we shall make it also.

Our notation for roulette lotteries will be the following: $(f_1B_1, \cdots, f_kB_k)$ denotes the roulette lottery that yields you B_1 with chance f_1, B_2 with chance f_2, and so on, where the f's sum to 1 and the B's stand for prizes from $\mathfrak{A}$ or for lottery tickets of any sort under consideration. In respect of preferences, we shall regard B as equivalent to $(1B)$, for any B.

Let $\mathcal{R}$ be the set of all (simple or compound) roulette lotteries with prizes in $\mathcal{A}$. We suppose that you have a preference ordering on $\mathcal{R}$ satisfying the axioms of utility theory (see for example [9]). Then it is possible to define a utility function u on $\mathcal{R}$ with the usual properties, including the following:

(i) Associated with each prize A is a number $u(A)$, called the utility of A.

(ii) If $B_1, \cdots, B_k$ are some prizes chosen from $\mathcal{A}$, the simple roulette lottery $(f_1B_1, \cdots, f_kB_k)$ has utility $f_1u(B_1) + \cdots + f_ku(B_k)$.

(iii) Your preference ordering on $\mathcal{R}$ is isomorphic with the numerical ordering of the corresponding utilities. We may normalize the utility function so that $u(A_1) = 1$, $u(A_0) = 0$. For the proofs of these statements we refer the reader to Luce and Raiffa.

Turning now to the horse race, let us assume that it has exactly s mutually exclusive and exhaustive possible outcomes, denoted by $h_1, \cdots, h_s$. For any s roulette lotteries $R_1, \cdots, R_s$, chosen from $\mathcal{R}$, the symbol $[R_1, \cdots, R_s]$ will denote the (compound) horse lottery that yields you R_1 if the outcome of the race is h_1, R_2 if the outcome is h_2, and so on. The set of all such horse lotteries will be denoted by $\mathcal{H}$. Let $\mathcal{R}^*$ denote the set of all roulette lotteries *whose prizes are such horse lotteries*, i.e. members of $\mathcal{H}$ instead of members of $\mathcal{A}$.

The essential device in our approach to subjective probability is to apply utility theory twice over, and then connect the two systems of preferences and utilities. We have just supposed that you have a preference ordering on $\mathcal{R}$, yielding the utility function u on $\mathcal{R}$. Let us now suppose that you also have a preference ordering on $\mathcal{R}^*$, satisfying the axioms of utility theory, with $\mathcal{H}$ taking the place of $\mathcal{A}$. We shall distinguish these preferences by starred symbols ($\succsim^*$, $\succ^*$), and denote the resulting utility function on $\mathcal{R}^*$ by u^*. We connect the two systems of preferences and utilities by the following two additional assumptions:

ASSUMPTION 1 (Montonicity in the prizes). *If* $R_i \succsim R_i'$, *then*

$$[R_1, \cdots, R_i, \cdots, R_s] \succsim^* [R_1, \cdots, R_i', \cdots, R_s].$$

ASSUMPTION 2 (Reversal of order in compound lotteries).

$$(f_1[R_1^1, \cdots, R_s^1], \cdots, f_k[R_1^k, \cdots, R_s^k]) \sim^* [(f_1R_1^1, \cdots, f_kR_1^k), \cdots, (f_1R_s^1, \cdots, f_kR_s^k)].$$

Assumption 1 says that if two horse lotteries are identical except for the prizes associated with one outcome, then your preference between the lotteries is governed by your preference between the prizes associated with that outcome. It is very much akin to Luce and Raiffa's "substitutability" assumption. Assumption 2 says that if the prize you receive is to be determined by both a horse race and the spin of a roulette wheel, then it is immaterial whether the wheel is spun before or after the race. This is akin in spirit to Luce and Raiffa's "reduction of compound lotteries" assumption (sometimes called the "algebra of combining" assumption), but is even more plausible than the latter. Here the "joy in gambling" is not abstracted away, and the chances f_i are not combined in any way. Assumptions 1 and 2 reflect the intuitive idea that the outcome of the horse race is not affected by—nor does it affect—any spin of the roulette wheel.

4. Existence of subjective probabilities. From Assumption 1 it follows that $[A_1, \cdots, A_1]$ is the most desired, and $[A_0, \cdots, A_0]$ the least desired horse lottery. Let us normalize the utility function u^* on $\mathcal{R}^*$ so that $u^*[A_1, \cdots, A_1] = 1$ and $u^*[A_0, \cdots, A_0] = 0$.

THEOREM. *There is a unique set of s non-negative numbers $p_1, \cdots, p_s$ summing to* 1, *such that for all $[R_1, \cdots, R_s]$ in $\mathcal{H}$,*

$$u^*[R_1, \cdots, R_s] = p_1 u(R_1) + \cdots + p_s u(R_s).$$

The number p_i is called the *subjective probability* of the outcome h_i of the race. From the theorem it follows that $u^*[R, \cdots, R] = u(R)$, so that we could identify $[R, \cdots, R]$ with R, and think of the preferences as constituting a single ordering defined on all lotteries simultaneously.

To prove the theorem, note first that by Assumption 1 any horse lottery is determined up to indifference by the utilities u of its entries; so that by abusing our symbolism slightly we can write our horse lotteries in the form $[r_1, \cdots, r_s]$, where $r_i = u(R_i)$. In particular, $u^*[1, \cdots, 1] = u^*[A_1, \cdots, A_1] = 1$ and $u^*[0, \cdots, 0] = u^*[A_0, \cdots, A_0] = 0$. Now define $p_i = u^*[0, \cdots, 1, \cdots, 0]$, where the 1 on the right side appears in spot i, and the other spots have 0.

LEMMA. *If for some $k > 0$ and for $i = 1, \cdots, s$, we have $0 \leqq r_i \leqq 1$ and $0 \leqq kr_i \leqq 1$, then $u^*[kr_1, \cdots, kr_s] = ku^*[r_1, \cdots, r_s]$.*

To prove this, first assume $k \leqq 1$. Then by Assumption 2 and the expected utility property for utilities on $\mathcal{R}$, we have

$$[kr_1, \cdots, kr_s] = [kr_1 + (1-k)0, \cdots, kr_s + (1-k)0] \\ \sim^* (k[r_1, \cdots, r_s], (1-k)[0, \cdots, 0]).$$

Hence

$$u^*[kr_1, \cdots, kr_s] = ku^*[r_1, \cdots, r_s] + (1-k)u^*[0, \cdots, 0] = ku^*[r_1, \cdots, r_s].$$

Next, if $k > 1$, it follows from the first half that

$$u^*[r_1, \cdots, r_s] = u^*[kr_1/k, \cdots, kr_s/k] = (1/k)u^*[kr_1, \cdots, kr_s];$$

and multiplying through by k, we complete the proof of the lemma.

Returning to the proof of the theorem, we set $c = r_1 + \cdots + r_s$. If $c = 0$, then all the $r_i = 0$, and the theorem is trivial. If $c > 0$, then the r_i/c are non-negative and sum to 1, and hence from Assumption 2 we conclude that

$$[r_1/c, \cdots, r_s/c] \sim^* ((r_1/c)[1, 0, \cdots, 0], \cdots, (r_s/c)[0, \cdots, 0, 1]).$$

Hence by the lemma (with $k = 1/c$) and the expected utility property, we have

$$\begin{aligned} u^*[R_1, \cdots, R_s] &= u^*[r_1, \cdots, r_s] \\ &= u^*[cr_1/c, \cdots, cr_s/c] = cu^*[r_1/c, \cdots, r_s/c] \\ &= c((r_1/c)u^*[1, 0, \cdots, 0] + \cdots + (r_s/c)u^*[0, \cdots, 0, 1]) \\ &= r_1 p_1 + \cdots + r_s p_s = p_1 u(R_1) + \cdots + p_s u(R_s), \end{aligned}$$

and the proof of the theorem is complete.

If our construction of subjective probabilities is applied to a set of exclusive and exhaustive outcomes h_i of some trial, such that each outcome has a known chance f_i, the "horse lotteries" degenerate into "roulette lotteries." (Formally this means that we are assuming $[R_1, \cdots, R_s] \sim (f_1R_1, \cdots, f_sR_s)$.) We now see at once from our definition of p_i that $p_i = f_i$ for all i. Thus in this case the subjective probability of any outcome is equal to the chance associated with that outcome. Since the two are equal, it does not matter much which word or symbol we use. The chance refers to the phenomenon, the probability refers to your attitude towards the phenomenon, and they are in perfect agreement.

Reverting now to proper horse lotteries for which there are no chances, we can interpret our construction of probabilities thus: p_i is equal to the chance such that you would as soon opt for a dollar contingent on the ith horse's winning as for a dollar with that chance. A priori this chance might change if we substituted two dollars for a dollar, but our theorem says that it does not.

To provide an adequate basis for the study of scientific inference, the above development of probability needs some extension. Horses must be translated into hypotheses and lotteries into decisions, and the concept of observation must be introduced. Though this involves no particular mathematical difficulty, it does raise some philosophical questions. We do not pursue the matter here.

The above theory of horse lotteries can be extended from a finite set of outcomes $\{h_i\}$ to a measurable space of outcomes. We refrain from going into details.

5. Comparison with the literature. Ramsey [13], de Finetti [4], Koopman [7], Good [5], Savage [14], Davidson and Suppes [3], [15], Kraft, Pratt and Seidenberg [8] and others have given definitions of subjective probability. Most often this has been done without reference to the physical theory of chances. Some writers, no doubt, have felt that because physics bristles with philosophical difficulties it is undesirable to base a logical concept on a physical concept; and that once a logical concept of probability has been adequately defined there is no need to contemplate any separate physical concept of chance. But it is possible to hold an opposite view, that probability is even obscurer than chance and that progress should preferably be from the more familiar to the less familiar, rather than the other way round. At the least, this route deserves to be explored. Blackwell and Girshick [1], Chernoff [2], Raiffa and Schlaifer [12] and others have explored it and obtained results closely related to those given here. The novelty of our presentation, if any, lies in the double use of utility theory, permitting the very simple and plausible assumptions and the simple construction and proof.

A comparison of our terminology and approach with Savage's [14] may be helpful. Our "horse lottery" corresponds to his "act"; our "outcome of the race" to his "state of the world"; our "prize" to his "consequence." Of Savage's six postulates, which he numbers P1 through P6, we share with him explicitly P1 (ordering of the horse lotteries—it is among the assumptions of utility theory). We also share P5, which Suppes [15] terms a structure axiom rather than a rationality axiom; it is our assumption that $A_1 > A_0$. P2 and P3 (the "surething" principle) are represented by our Assumption 1, which is close to P3. P4 can be translated to our context by saying that your guess about which horse will win is not affected by the size of the prize offered for each horse. Inherent in the intuitive background of P4 is the assumption that you explicitly guess which horse will win, and that you are able and willing to judge which of two horses has a higher probability of winning. Thus P4 implicitly assumes the existence of comparative probabilities; rather than constructing subjective probabilities from preferences, Savage has constructed numerical subjective probabilities from comparative subjective probabilities. There is nothing corresponding to P4 in our approach. P6 is a continuity postulate which plays a role in defining subjective probability similar to that of the continuity (or "Archimedean") assumption of utility theory in defining the utility function. Here again there is implicit use of the probability notion; but now we have not only comparative probabilities, but also the existence of probabilities that are "small." Although there is nothing directly corresponding to P6 in our approach, we may say, very roughly, that there is no need for such an assumption in our approach because we start out with chances, which have a continuous range.

It would seem that a postulate asserting the existence of comparative probabilities, like Savage's P4, occurs in many presentations of subjective probability. For example, the requirement L_3 of Blackwell and Girshick [1], which is rather sophisticated and includes Savage's P2 as a special case, includes also as a special case something similar in spirit to Savage's P4, though not identical with it. We feel that the preference relation is a step closer to experience than the comparative probability relation, and that it is desirable to base subjective probabilities on the preference relation only, without any a priori reference—implicit or explicit—to comparative probabilities.

In this respect our treatment resembles that of Raiffa and Schlaifer [12]. They, however, claim only that their treatment is "*very* informal". It appears to us that they make use of what we call Assumption 2 without mentioning it. In this paper, Assumption 2 is certainly not a tautology, as can easily be shown by counterexample. However, in a different context, with different basic assumptions, our Assumption 2 *could* be a tautology. It is indeed very obvious, intuitively.

To conclude, we remark that anyone who wishes to avoid a concept of physical chance distinct from probability may reinterpret our construction as a method of defining more difficult probabilities in terms of easier ones. Such a person may consider that probabilities can be assigned directly to the outcomes of spins of a roulette wheel, flips of a coin, and suchlike, from considerations of symmetry. The probabilities may be so widely agreed on as to be termed impersonal or objective probabilities. Then, with some assumptions concerning independence, our construction can be used to define subjective probabilities for other sorts of outcomes in terms of these objective probabilities.

In fact, apropos of the above remark and of the preceding comments on comparative probabilities, Professor Savage has pointed out in private correspondence that the incidence of P4 in the axiom system of [14] can be weakened, so that P4 is asserted only for a subset of acts, corresponding (roughly) to our roulette lotteries. He has then deduced the original P4. Thus an unrestricted postulate of comparative probabilities is not essential to the derivation of numerical probabilities from Savage's viewpoint.

6. Acknowledgments. We are indebted to Professor Savage for much valuable discussion, and to the referee for the reference to Chernoff.

Note added in proof. Professor T. S. Ferguson has pointed out that our application of utility theory to horse lotteries in Section 4 depends on the assumption of strict preference,

$$[A_1, \cdots, A_1] >^* [A_0, \cdots, A_0].$$

This would follow at once from our previous assumption that $A_1 > A_0$ if we made the identification of $[R, \cdots, R]$ with R.

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