Rooftop Theorem for Concave functions

This theorem asserts that if $f$ is a differentiable concave function of a single variable, then at any point $x$ in the domain of $f$, the tangent line through the point $(x, f(x))$ lies entirely above the graph of $f$. You should draw a picture.

**Theorem 1.** If $f$ is a continuously differentiable concave function of a single variable, defined on a real interval $I$, then for all $x_1$ and $x_2$ in $I$,

$$f(x_1) + (x_2 - x_1)f'(x_1) \geq f(x_2).$$

Geometrically, this theorem says that the tangent line to the graph of $f$ passing through any point $(x_1, f(x_1))$ must lie entirely on or above the graph of $f$. You should draw a couple of pictures to convince yourself of this geometry.

**Proof.** Since $f$ is a concave function, it must be that for all $x_1$ and $x_2$ in $I$, and all $t \in [0,1]$,

$$f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2). \quad (1)$$

Rearranging terms, we see that Equation 1 is equivalent to

$$f(x_1 + t(x_2 - x_1)) - f(x_1) \geq t(f(x_2) - f(x_1)). \quad (2)$$

Dividing both sides of equation 2 by $t$, we have

$$\frac{f(x_1 + t(x_2 - x_1)) - f(x_1)}{t} \geq f(x_2) - f(x_1) \quad (3)$$

This implies that

$$(x_2 - x_1) \frac{f(x_1 + t(x_2 - x_1)) - f(x_1)}{t(x_2 - x_1)} \geq f(x_2) - f(x_1) \quad (4)$$

Then it must be that

$$(x_2 - x_1) \lim_{t\to 0} \frac{f(x_1 + t(x_2 - x_1)) - f(x_1)}{t(x_2 - x_1)} \geq f(x_2) - f(x_1) \quad (5)$$

But then we have

$$\lim_{t\to 0} \frac{f(x_1 + t(x_2 - x_1)) - f(x_1)}{t(x_2 - x_1)} = \lim_{h\to 0} \frac{f(x_1 + h) - f(x_1)}{h} = f'(x_1) \quad (6)$$

It follows that:

$$(x_2 - x_1)f'(x_1) \geq f(x_2) - f(x_1). \quad (7)$$

Rearranging Equation 8, we have the desired result, namely

$$f(x_1) + (x_2 - x_1)f'(x_1) \geq f(x_2). \quad (8)$$

$\Box$
Now an easy and important consequence of the Rooftop Theorem is the following.

**Theorem 2.** If \( f \) is a continuously differentiable function of a single variable, defined on a real interval \( I \), then \( f \) is a concave function if and only if \( f''(x) \leq 0 \) for all \( x \in I \).

One proof of this theorem is to apply Taylor’s theorem and the Rooftop theorem. (Hint: Write the exact form of the second order Taylor’s expansion.)

Here is another proof. Suppose that \( f \) is a concave function. Choose any two points \( x \) and \( y \) in \( I \) such that \( x > y \). The Rooftop Theorem implies that \( f(x) - f(y) \leq f'(y)(x - y) \) and also \( f(y) - f(x) \leq f'(x)(y - x) \). The second inequality is equivalent to \( f(x) - f(y) \geq f'(x)(x - y) \). It follows that \( f'(x)(x - y) \leq f(x) - f(y) \leq f'(y)(x - y) \) and hence that \( f'(x) \leq f'(y) \) whenever \( x > y \). But this means that \( f' \) is a non-increasing function and hence \( f''(x) \leq 0 \) for all \( x \in I \).

A similar argument establishes the converse.