1) Elmer Kink’s utility function is \( \min \{x_1, 2x_2\} \).

Draw a few indifference curves for Elmer.
These are L-shaped, with the corners lying on the line \( x_1 = 2x_2 \).

Find each of the following for Elmer:

- **His Marshallian demand function for each good.**

  Elmer’s chosen bundle always lies on the point where his budget line meets a kink of the L. This means that his Marshallian demand satisfies the two equations \( x_1 = 2x_2 \) and \( p_1x_1 + p_2x_2 = m \). Solving these equations, we have \((2p_1 + p_2)x_2 = m\) and hence

  \[
  x_2(p_1, p_2, m) = \frac{m}{2p_1 + p_2}. \tag{1}
  \]

  Then

  \[
  x_1(p_1, p_2, m) = 2x_2(p_1, p_2, m) = \frac{2m}{2p_1 + p_2}. \tag{2}
  \]

- **His Indirect utility function.**

  His indirect utility function is

  \[
  v(p, m) = u(x_1(p, m), x_2(p, m)) = \frac{2m}{2p_1 + p_2}. \tag{3}
  \]

- **His Hicksian demand function for each good.** His Hicksian demand function for \( x_i \) can be found by differentiating the expenditure function, which we solve for in the next question. In particular, this function is

  \[
  x_1(p, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1} = u \tag{4}
  \]

  and

  \[
  x_2(p, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_2} = u/2 \tag{5}
  \]

- **His Expenditure function.**

  His expenditure function is \( e(p_1, p_2, u) = (2p_1 + p_2)u \). We can find this by noting that \( v(p, e(p, u)) = u \). Using the solution for \( v(p, u) \) found in Equation 3 we see that

  \[
  \frac{2e(p, u)}{2p_1 + p_2} = u.
  \]

  Rearranging terms, we have

  \[
  e(p, u) = (p_1 + \frac{p_2}{2})u. \tag{6}
  \]
• Verify that Roy’s Law applies in Elmer’s case. Roy’s law requires that

\[
x_i(p, m) = -\frac{\partial v(p_1, p_2, m)}{\partial p_i} - \frac{\partial v(p_1, p_2, m)}{\partial m}.
\]  

(7)

In this case,

\[
\frac{\partial v(p_1, p_2, m)}{\partial p_1} = -\frac{2m}{(2p_1 + p_2)^2}
\]  

(8)

and

\[
\frac{\partial v(p_1, p_2, m)}{\partial m} = -\frac{1}{(2p_1 + p_2)}.
\]  

(9)

Substituting from Equations 8 and 9 into Equation 7, we have

\[
x_1(p, m) = \frac{2m}{2p_1 + p_2}
\]  

(10)

This coincides with the solution for \( x_1(p, m) \) found in Equation 2. A similar argument shows that Roy’s law applies for Good 2.

2) Consider the function \( f(x_1, x_2) = ax_1 + bx_2 + cx_1^2 + dx_1x_2 + ex_2^2 \).

(a) Write down the Hessian matrix for this function.

The Hessian matrix is

\[
\begin{pmatrix}
2c & d \\
d & 2e
\end{pmatrix}
\]

(b) For what values of the parameters \( a, b, c, d \), and \( e \) is \( f \) a concave function?

We need \( c \leq 0, e \leq 0 \) and \( 4ce > d^2 \).

(c) For what values of these parameters is \( f \) a convex function?

\( c \geq 0, e \geq 0 \), and \( 4ce \geq d^2 \).

(d) For what values of these parameters is \( f \) neither concave, nor convex?

This will be the case if \( c \) and \( e \) are of opposite signs. It will also be the case if \( 4ce < d^2 \).

(e) For what values of these parameters is \( f \) both concave and convex?

This will be the case if \( c = d = e = 0 \), in which case the function is linear. A linear function is both concave and convex.

3) Prove that a concave function must be quasi-concave. Give an example of a quasi-concave function that is not concave. (Be sure to show that your example is quasi-concave and that it is not concave.)

To show that a function is quasi-concave, we must show that if \( f(x) \geq f(x') \), then \( f(tx + (1 - t)x') \geq f(x') \) for all \( t \) such that \( 0 \leq t \leq 1 \). Suppose that \( f \) is
a concave function and \( f(x) \geq f(x') \). Since \( f \) is a concave function, it must be that \( f(tx + (1-t)x') \geq tf(x)+(1-t)f(x') \) for all \( t \) such that \( 0 \leq t \leq 1 \), which means that \( f \) is a quasi-concave function.

Let \( f \) be the function of a single variable \( f(x) = x^3 \) defined on the real line. Then \( f''(x) = 3x^2 > 0 \) for all \( x \neq 0 \), so \( f \) cannot be a concave function. Suppose that \( f(x) \geq f(x') \). Then since \( f \) is an increasing function, it must be that \( x \geq x' \). Therefore for \( t \) such that \( 0 \leq t \leq 1 \), it must be that \( tx + (1-t)x' \geq x' \). Since \( f \) is an increasing function, it follows that \( f(tx + (1-t)x') \geq f(x') \). Therefore if \( f(x) \geq f(x') \), then \( f(tx+(1-t)x') \geq f(x') \), which means that \( f \) is quasi-concave.

4) Define each of the following without mentioning principle minors. In your definition, be sure to use a grammatically complete sentence with subject and predicate.

(a) A negative semi-definite matrix.

An \( n \times n \) matrix \( A \) is negative semi-definite if for every \( n \) dimensional vector \( z \), \( z^tAz \leq 0 \) where \( z^tAz \) is the quadratic form,

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} z_i A_{ij} z_j.
\]

(b) A positive semi-definite matrix An \( n \times n \) matrix \( A \) is positive semi-definite if for every \( n \) dimensional vector \( z \), \( z^tAz \geq 0 \) where \( z^tAz \) is the quadratic form,

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} z_i A_{ij} z_j.
\]

(c) A matrix that is neither positive nor negative semi-definite. A matrix \( A \) is neither positive semi-definite nor negative semi-definite if there exists a vector \( x \) such that \( x^tAx > 0 \) and another vector \( y \) such that \( y^tAy < 0 \).

(d) How can one use principle minors to test whether a matrix is negative semi-definite? positive semidefinite? A matrix is negative semi-definite if and only if all principle minors of size \( k \) are non-positive for all odd \( k \) and non-negative for all even \( k \). A matrix is positive semi-definite if and only if all principle minors are non-negative. (Incidentally, to test for positive definiteness or negative definiteness, one need only look at leading principle minors. But the conditions for semi-definiteness require attention to all principle minors. I will post a couple of pages from Blume and Simon that explain this situation.)

5) Dracula, the mortgage broker, is an expected utility maximizer, with von Neumann-Morgenstern utility function

\[
u(x) = \frac{1}{2} \sqrt{x}.
\]
A) Dracula currently holds a portfolio of subprime mortgages, all in the same town. If the local economy goes bad, these mortgages will be worthless and his wealth will be zero. If the local economy does not go bad, his wealth will be $1,000,000. The probability that the local economy will go bad is 1/10. Calculate the certainty equivalent of Dracula’s current holdings.

Where $CE$ is the certainty equivalent of Dracula’s portfolio, it must be that

$$\frac{1}{2} \sqrt{CE} = .1 \times \frac{1}{2} \sqrt{0} + .9 \times \frac{1}{2} \sqrt{1,000,000}.$$ 

Double and square both sides of this equation to find that $CE = 810,000$.

B) Suppose that Dracula can buy mortgage insurance such that he will pay $X$ to the insurer if the mortgages do not fail and he will receive a payment of $1,000,000 − X$ from the insurer if the mortgages fail. What is the largest amount $X$ that Dracula would be willing to pay for this insurance.

Notice that if he buys the insurance, his portfolio is certain to be worth $1,000,000 − X$. As we saw in part A, if he does not buy the insurance, the expected utility of his portfolio is the same as that of having 810,000 for sure. He will be willing to buy the insurance, so long as $1,000,000 − X > 810,000$, which means that the most he would be willing to pay for this insurance is $190,000.

C) What is the largest amount that Dracula would be willing to pay for the insurance if his von Neumann-Morgenstern utility function were linear in his wealth?

In this case, he would simply want to maximize his expected wealth. His expected wealth without insurance is $900,000. If he buys the insurance for $X$, he will be sure to have wealth of $1,000,000 − X$. Therefore the most he would be willing to pay is $100,000.

D) Suppose that in addition to his risky mortgage portfolio, Dracula has safe assets that are worth $V$, no matter what happens. Write an expression for the certainty equivalent of Dracula’s assets (as a function of $V$), assuming he does not buy insurance. Write an expression for the largest amount $X$ that Dracula would be willing to pay for the insurance described in Part B (as a function of $V$).

Where $CE$ is the certainty equivalent, we have

$$\frac{1}{2} \sqrt{CE} = \frac{1}{10} \frac{1}{2} \sqrt{V} + \frac{9}{10} \frac{1}{2} \sqrt{V + 1,000,000}.$$ 

Doubling and squaring both sides of the equation, we have

$$CE = \left( \frac{1}{10} \sqrt{V} + \frac{9}{10} \sqrt{V + 1,000,000} \right)^2.$$
If Dracula buys the insurance at price $X$, his wealth is sure to be $V + 10,000,000 - X$. He will buy the insurance only if this wealth is at least as large as the certainty equivalent of his current portfolio. That is, if and only if the price $X$ of the insurance is no larger than

$$V + 10,000,000 - \left( \frac{1}{10} \sqrt{V} + \frac{9}{10} \sqrt{V + 1,000,000} \right)^2.$$