d. Suppose preferences are locally insatiable and x solves the CP, but $p \cdot x < y$. Let $\beta = y - p \cdot x$, $\gamma = \max\{p_i; i = 1, \ldots, k\}$, and $\epsilon = \beta/(k\gamma)$. By local insatiability, some x' within ϵ of x is strictly preferred to x; that is, $u(x') > u(x)$. But $p \cdot x' = \sum_{i=1}^{k} p_i x_i' \leq \sum_{i=1}^{k} p_i(x_i + \epsilon) \leq p \cdot x + k\gamma\epsilon = p \cdot x + \beta = y$. That is, x' is affordable at prices b with wealth y, and it is strictly preferred to x, contradicting the supposed optimality of x for prices p and income y. ▪

A simple picture goes with the proof of part d. See Figure 3.3 and the accompanying caption.

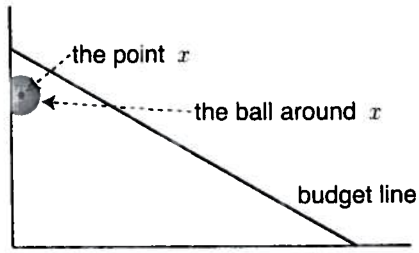

Figure 3.3. The picture for the proof of Proposition 3.1d. If $p \cdot x < y$, then x lies "below" the budget line, and we can put a ball of strictly positive radius around x such that $p \cdot x' \leq y$ for every x' inside the ball. Local insatiability ensures that no matter how small the radius of this ball, as long as it is strictly positive, some (nonnegative) point inside the ball must be strictly better than x, which contradicts the optimality of x at the given prices and income.

3.3. The Marshallian Demand Correspondence and Indirect Utility Function

Fix the utility function u for a particular consumer. Assume u is continuous, reflecting this consumer's continuous preferences. For each set of strictly positive prices p and nonnegative income level y, we have a version of the CP for this consumer. By virtue of Proposition 3.1b, we know that the CP has a solution. Let $\mathbf{D}(p, y)$ denote the set of solutions for the fixed u, as a function of p and y, and let $\nu(p, y)$ denote the value of the optimal solution; that is, $\nu(p, y) = u(x^*)$ for any $x^* \in \mathbf{D}(p, y)$. ($\mathbf{D}$ is a mnemonic for *demand*.)

Definition 3.2. *Fixing u, the set $\mathbf{D}(p, y)$ is called* ***Marshallian demand*** *at prices p and income y, and the correspondence $(p, y) \Rightarrow \mathbf{D}(p, y)$ is called the* ***Marshallian demand correspondence****. The number $\nu(p, y)$ is called the* ***indirect utility*** *at p and y, and the function $(p, y) \rightarrow \nu(p, y)$ is called the* ***indirect utility function****.*

Proposition 3.3 (Berge's Theorem applied to the consumer's problem).

a. *For all $p \in R^k_{++}$, $y \geq 0$, and $\lambda > 0$, $\mathbf{D}(p, y) = \mathbf{D}(\lambda p, \lambda y)$ and $\nu(p, y) = \nu(\lambda p, \lambda y)$.*

b. *The Marshallian demand correspondence is upper semi-continuous. If, for some open set of prices and income, Marshallian demand is singleton valued (that is, the CP has a* *unique solution for all price-income pairs inside that open set), then the function that describes the solution as a function of (p, y) is a continuous function.*

c. *The indirect utility function is continuous.*

If the notion of a correspondence or an upper semi-continuous correspondence is new to you, or if you have never heard of Berge's Theorem, also known as the Theorem of the Maximum, please consult Appendix 4 before attempting to understand either the statement of this proposition or its proof.

Proof. Part a of the proposition is a simple corollary of Proposition 3.1a. As for parts b and c, these come from a straightforward application of Berge's Theorem, given in Appendix 4 as Proposition A4.7. Since this is our first application of this important general result, I spell out the details: The consumer's problem is a parametric constrained maximization problem

$$\text{maximize } u(x), \quad \text{subject to } x \in \mathbf{B}(p, y).$$

The variable in the problem is x, and the parameter is the vector (p, y). Berge's Theorem tells us that the solution correspondence is nonempty valued if, for each set of parameters, the constraint set is nonempty and compact and the objective function is continuous in the variables; moreover, the solution-set correspondence is upper semi-continuous and the value-of-the-solution function is continuous, both in the parameters, as long as the objective function is jointly continuous in the variables and the parameters and the constraint-set correspondence is locally bounded and continuous in the parameters. (The version of Berge's Theorem given in Appendix 4 is somewhat more robust than this simple rendition, but the simple rendition is adequate for now.)

In this particular application, the objective function is independent of the parameters and assumed continuous in the variables. Therefore, the objective function presents no problem. We already showed that each $\mathbf{B}(p, y)$ is nonempty and compact. So once we show that $(p, y) \rightrightarrows \mathbf{B}(p, y)$ is a continuous and locally bounded correspondence, the conditions of Berge's Theorem are met, and its conclusions are established. To begin with local boundedness, fix a pair (p, y), and let

$$\epsilon = \frac{1}{3} \min_{i=1,\ldots,k} p_i.$$

For all (p', y') within ϵ of (p, y), $x \in \mathbf{B}(p', y')$ must solve $p' \cdot x \leq y'$. Since in the sum $p' \cdot x$ each term is nonnegative and $y' \leq y + \epsilon$, this inequality implies that $p'_i x_i \leq (y + \epsilon)$; hence $x_i \leq (y + \epsilon)/p'_i$. But $p'_i \geq p_i - \epsilon \geq \frac{2}{3} p_i \geq 2\epsilon$; therefore $x_i \leq (y + \epsilon)/(2\epsilon)$, which provides a uniform bound for $x \in \mathbf{B}(p', y')$.

Continuity of the constraint correspondence is shown by proving separately that the correspondence is upper and lower semi-continuous. To show upper semi-continuity (having shown local boundedness), we take a sequence $\{(x^n, p^n, y^n)\}$

with $x^n \in \mathbf{B}(p^n, y^n)$ for each n and with limit $\{(x, p, y)\}$.[2] (We are using superscripts here because subscripts denote components of the vectors x and p.) Of course, $x^n \geq 0$ for each n, and since the positive orthant is closed, this implies that $x \geq 0$. Moreover, $p^n \cdot x^n \leq y^n$ for each n; using continuity of the dot product, this implies that the limit of the left-hand side, $p \cdot x$, is less than or equal to the limit of the right-hand side, y. Therefore, $p \cdot x \leq y$ and $x \in \mathbf{B}(p, y)$. This establishes upper semi-continuity.

To show lower semi-continuity, for each sequence $\{(p^n, y^n)\}$ with limit (p, y) and a point $x \in \mathbf{B}(p, y)$, we must produce a sequence $\{x^n\}$ with limit x and such that $x^n \in \mathbf{B}(p^n, y^n)$ for each n. If $y = 0$, then $x = 0$ (prices are strictly positive); therefore $x^n = 0$ for all n works. If $x = 0$, the same choice of x^n will do. Therefore, we can assume that $y > 0$, $x \neq 0$ and, by going far enough out in the sequence, that the $p^n \cdot x$ are uniformly bounded away from zero. Let

$$x^n = \frac{y^n}{y} \frac{p \cdot x}{p^n \cdot x} x.$$

Since $y^n \to y$ and $p^n \to p$, continuity of the dot product implies that $x^n \to x$. It remains to show that $x^n \in \mathbf{B}(p^n, y^n)$. Nonnegativity of x^n is no problem, since x^n is just a scale copy of x. Moreover,

$$p^n \cdot x^n = \frac{y^n}{y} \frac{p \cdot x}{p^n \cdot x} p^n \cdot x = \frac{y^n}{y} p \cdot x \leq y^n.$$

■

3.4. Solving the CP with Calculus

When economists build models populated by consumers, it is common practice to specify the individual consumer's utility function and to solve the CP analytically, using calculus. (It is also common to begin directly with the consumer's demand function, or even with a demand function that aggregates the demands of a population of consumers. We discuss these alternative practices in later chapters.) To build and work with such models, you must be able to carry out this sort of analytical exercise. In this section, we discuss how this is done, and (more important to future developments) how to interpret pieces of the exercise.

The CP is a problem of constrained optimization: A numerical objective function (utility) is to be maximized, subject to some inequality constraints (the budget constraint, and all variables nonnegative). Assuming the objective function and constraint functions are differentiable and otherwise well behaved, the standard theory of constrained optimization establishes necessary and sufficient conditions

[2] In proving upper and lower semi-continuity, we look at sequences of parameters—in this case, sequences $\{p^n, y^n\}$—that converge to points in the domain of the correspondence. Therefore, the limit price vector p here must be strictly positive.