Name ______________________________

Final Exam, Economics 210A, December 2014

Answer any 7 of these 8 questions Good luck!

1) For each of the following statements, state whether it is true or false. If it is true, prove that it is true. If it is false, show that is false by providing a counterexample. Your answers should include a definition of each of the technical terms used.

A) If preferences are strictly monotonic, they must be locally non-satiated.

True. Preferences on $\Re^n$ are strictly monotonic if $x > y$ implies that $x \succ y$. Preferences are locally non-satiated if for all $x \in \Re^n_+$, and for all $\epsilon > 0$ there exists $x' \in \Re^n_+$ such that $|x' - x| < \epsilon$ and $x' \succ x$. If preferences are strictly monotonic, then for any $x = (x_1, \ldots, x_n)$, let $y = (x_1 + \epsilon/2, x_2, \ldots, x_n)$. Then $|y - x| < \epsilon$ and since preferences are strictly monotonic, $y \succ x$.

B) If preferences are locally non-satiated, they must be strictly monotonic.

False. Consider the preference relation $\succeq$ on $\Re^2_+$ defined by $(x_1, x_2) = x_1 + (x_2 - 1)^2$. This function is locally non-satiated, but it is not strictly monotonic.

C) If preferences are transitive, convex, and continuous, they must be locally non-satiated.

False. (I leave it to you to write down definitions of transitive, convex, and continuous preferences. Consider the preference relation $\geq$ on $\Re^2_+$ such that for all x and y in $\Re^2_+$, $x \succeq y$. This relation defines preferences that are transitive, convex, and continuous, but is clearly not locally non-satiated.

D) If preferences are non-satiated and strictly convex, they are locally non-satiated.

True. Preferences are non-satiated if for any x there exists y such that $y \succ x$. If preferences are strictly convex, then if $y \succ x$, then for all $\lambda \in (0, 1)$,

$\lambda y + (1-\lambda)x \succ x$. Since this is true for arbitrarily small $\lambda > 0$, it must be that for any $\epsilon > 0$, there exists $x' = \lambda y + (1-\lambda)x$ such that $x' \succ x$ and $|x' - x| < \epsilon$.

2) Consider a consumer with preference relation $\succeq$ over an n-dimensional commodity space. This consumer has a utility function $u(x)$ and a Marshallian demand correspondence $X(p,m)$. Let $e(p,u)$ be the expenditure function for this consumer.

A) Let $x \in X(p,m)$. If the preference relation $\succeq$ is locally non-satiated, prove that $px = m$.

If $x \in X(p,m)$ then it must be that $px \leq m$ and if $x' \succ x$, then $px' > m$. Suppose that $x \in X(p,m)$ and $px < m$. We can always find $\epsilon > 0$ small enough so that for all y such that if $|x - y| < \epsilon$, then $py < m$. If there is local nonsatiation, then for this $\epsilon 0$, there must be an x' such that $|x' - x| < \epsilon$ and $x' \succ x$. But then we have x' such that $px' < m$ and $x' \succ x$. Therefore it cannot be that $x \in X(p,m)$. It follows that if $x \in X(p,m)$, then $px = m$.

B) Give an example where $\succeq$ is not locally non-satiated and where $x \in X(p,m)$ and $px < m$.

Suppose that $x \succeq y$ for all x and y in $\Re^n_+$ and let $p >> 0$ and $m = 1$. Then all x such that $px \leq 1$ are contained in $X(p,m)$. But not all such points have $px = 1$.

C) What assumptions do you need in order to prove that if $x \in X(p,m)$, then $e\,(p, u(x)) \leq px$? Give a proof, using whatever assumptions are needed.

We don't need any more assumptions to prove this. By definition, $e(p, u(x))$ is the minimum of px' such that $u(x') \geq u(x)$. Clearly this minimum cannot be larger than px.

D) Prove that if $x \in X(p,u)$, then $e\,(p, u(x)) = px$. Use whatever assumptions are needed and show that it takes stronger assumptions to prove this result than to prove the result in Part C.

Let us assume that preferences are locally non-satiated. Let $x \in X(p,m)$. Then from Part A, we know that $px = m$. From Part C, we know that $e(p, u(x)) \leq px$. Suppose that $e(p, u(x)) < px$. Then $e(p,u) = px'$ where $u(x') \geq u(x)$ and $px' < px = m$. But from Part A, we know that this cannot be the case. This proves that if preferences are locally non-satiated, and $x \in X(p, m)$, then $e\left(p, u(x)\right) \leq px$.

We need something like local non-satiation to prove this result. For example, if the consumer is indifferent among all bundles, it is easy to see that the result is not true.

3) A consumer consumes two goods. For each of the following utility functions, find this consumer's indirect utility function and this consumer's expenditure function.

A) $U(x_1, x_2) = x_1 + x_2$.

$$v(p_1, p_2, m) = \frac{m}{\min\{p_1, p_2\}}$$

$$e(p_1, p_2, u) = \min p_1, p_2 u.$$

B) $U(x_1, x_2) = \min\{x_1, x_2\}$.

$$v(p_1, p_2, m) = \frac{m}{p_1 + p_2}.$$

$$e(p_1, p_2, u) = (p_1 + p_2)u.$$

C) $U(x_1, x_2) = x_1 + \ln x_2$.

$$v(x_1, x_2) = \frac{m}{p_1} - 1 + \ln p_1 - \ln p_2$$

if $m \geq p_1$.

$$v(p_1, p_2, m) = \ln m - \ln p_2$$

if $m < p_1$.

$$e(p_1, p_2, u) = p_1 u - p_1 \ln p_1 + p_1 \ln p_2 + p_1$$

if $u \geq \ln p_1 - \ln p_2$

$$e(p_1, p_2) = p_2 e^u$$

if $u < \ln p_1 - \ln p_2$.

D) $U(x_1, x_2) = x_1 x_2$

$$V(p_1, p_2, m) = \frac{m^2}{4p_1p_2}.$$
$$e(p_1, p_2, u) = 2\sqrt{p_1p_2m}.$$

4) A consumer consumes two goods. For each of the following utility functions, find this consumer's indirect utility function and this consumer's expenditure function.

A) $U(x_1, x_2) = \ln(x_1 + x_2)$

$$v(p_1, p_2, m) = \ln\left(\frac{m}{\min\{p_1, p_2\}}\right)$$

$$e(p_1, p_2, u) = \min\{p_1, p_2\}e^u.$$

B) $U(x_1, x_2) = \ln(\min\{x_1, x_2\})$

$$v(p_1, p_2, m) = \ln\left(\frac{m}{p_1 + p_2}\right)$$
$$e(p_1, p_2, u) = (p_1 + p_2)e^u.$$

C) $U(x_1, x_2) = \ln(x_1 + \ln x_2)$

$$v(p_1, p_2, m) = \ln\left(\frac{m}{p_1} - 1 + \ln p_1 - \ln p_2\right)$$

if $m \geq p_1$

$$e(p_1, p_2, u) = p_1e^u + p_1 - p_1\ln p_1 + p_1\ln p_2$$

if $u \geq \ln(\ln p_1 - \ln p_2)\, v$.

$$v(p_1, p_2, m) = \ln(\ln m - \ln p_2)$$

if $m < p_1$.

$$e(p_1, p_2, m) = p_2e^{e^u}$$

if $u < \ln(\ln p_1 - \ln p_2)$.

5) A consumer consumes two goods. His preferences over lotteries of consumption bundles are represented by a von Neumann Morgenstern expected utility function.

A) Suppose that this consumer's preferences over lotteries of consumption bundles are represented by the expected value of $U(x_1, x_2) = \ln(x_1 + x_2)$. If this consumer's income is sure to be m, would she prefer that the price vector be $(2, 2)$ with certainty, or that the price vector will be $(1, 3)$ with probability $1/2$ and $(3, 1)$ with probability $1/2$ or would she be indifferent? Explain your answer.

Taking note of the answer to Problem 4, we find that the expected utility will be the expected value of the lottery will be the expected value of her indirect utility, which will be

$$1/2 \ln\left(\frac{m}{\min\{1,3\}}\right) + 1/21/2 \ln\left(\frac{m}{\min\{3,1\}}\right) = \ln m.$$

Getting prices $(2, 2)$ with certainty would have expected utility would give her an expected utility of

$$\frac{m}{\min\{2,2\}} = \ln m/2.$$

So she would prefer the lottery.

B) Suppose that this consumer's preferences over lotteries of consumption bundles are represented by the expected value of $U(x_1, x_2) = \ln(\min\{x_1, x_2\})$. If this consumer's income is sure to be m, would she prefer that the price vector be $(2, 2)$ with certainty, or that the price vector will be $(1, 3)$ with probability $1/2$ and $(3, 1)$ with probability $1/2$ or would she be indifferent? Explain your answer.

With the lottery, her expected utility would be

$$1/2\left(\ln \frac{m}{1+3}\right) + 1/2\left(\ln \frac{m}{3+1}\right) = \ln \frac{m}{4}.$$

With the certain prospect, her expected utility would be

$$\ln \frac{m}{2+2} = \ln \frac{m}{4}$$

So she would be indifferent between the lottery and the sure thing.

6) Part A) Consider a pure exchange economy with two goods and 100 Type A consumers and 100 Type B consumers.

Type A's have preferences represented by the utility function

$$U^B(x_1, x_2) = x_1x_2.$$

Each type A has an initial endowments of 0 units of good 1 and 4 units of good 2. Type B's have preferences represented by the utility function

$$U^A(x_1, x_2) = x_1 + 2x_2^{1/2}.$$

Each type B has an initial endowment of 5 units of good 1 and 3 units of good 2. Let good 1 be the numeraire. Find a competitive equilibrium price for good 2. Also find the equilibrium consumption of each good for each consumer.

If the price of good 1 is 1, the equilibrium price of good 2 is $1/\sqrt{5}$. The equilibrium consumption of good 2 by Consumer A is 2 and the equilibrium consumption of good 2 by B is 5. The equilibrium consumption of good 1 by each Consumer of Type A is $2/\sqrt{5}$ and the equilibrium consumption of good 1 by each Consumer of Type B is $6 - 2/sqrt5$.

Part B) Consider a pure exchange economy with two goods and 100 Type A consumers and 100 Type B consumers. Type A's have preferences represented by the utility function

$$U^A(x_1, x_2) = x_1 + 2x_2^{1/2}.$$

Each type A has an initial endowments of 0 units of good 1 and 4 units of good 2. Type B's have preferences represented by the utility function

$$U^B(x_1, x_2) = x_1 + x_2^{1/2}.$$

Each type B has an initial endowment of 6 units of good 1 and 2 units of good 2. Let good 1 be the numeraire. Find a competitive equilibrium price for good 2. Also find the equilibrium consumption of each good for each consumer.

One possibility is that there is a competitive equilibrium price in which both consumers consume positive amounts of both goods. If this is the case, then at the equilibrium price, the type A's demands would satisfy the equation $x_2^A = 1/p^2$ and the type B's demands would satisfy $x_2^B = 1/4p^2$. Since the total supply of good 2 is 600 units, we would have supply equal to demand when

$$100\left(\frac{1}{p^2} + \frac{1}{4p^2}\right) = 600.$$

This equation is solved when $p = \sqrt{5/24}$. But at price $p = \sqrt{5/24}$, Type A consumers would choose a corner solution in which they demanded only good 2. So there can not be an equilibrium where both buy both goods.

Let us look for an equilibrium in which Type A's are at a corner solution, consuming only good 2 and type B's consume both goods. If Type A's consume only good 2, since they have no endowment of good 1, they must consume exactly their endowment of good 2. This means that in equilibrium, each type B consumes 6 units of good 1 and 2 units of good 2. This will happen when

$$\frac{1}{4p^2} = 2$$

which implies $p = \sqrt{(1/8)}$. So the competitive equilibrium price is $p = \sqrt{1/8}$ and the equilibrium consumptions are the same as the initial endowments. A little more checking shows that there are no other equilibria.

7) A poor fellow finds a lottery ticket that pays X ducats with probability p and 0 with probability $(1-p)$. This poor fellow has initial wealth of W_0 ducats, which he will be able to keep regardless of what the lottery ticket pays. Suppose that the poor fellow is an expected utility maximizer with von Neumann Morgenstern utility function

$$U(W) = \frac{1}{\gamma}W^\gamma$$

where W is his wealth.

A) Write down the certainty equivalent for the gamble that he faces if he keeps the lottery ticket. (This will be a function of X, p, and W_0.)

$$CE = (p(W_0 + x)^\gamma + (1-p)W_0^\gamma)^{1/\gamma}$$

B) For someone with the risk preferences of this poor fellow, calculate the Arrow Pratt measure of absolute risk aversion, as a function of his wealth. Also calculate his measure of relative risk aversion.

Coefficient of absolute risk aversion is

$$-\frac{U''(W)}{U'(W)} = \frac{1-\gamma}{W}$$

Coefficient of relative risk aversion is

$$-\frac{WU''(W)}{U'(W)} = 1-\gamma$$

C) If $X = 100$, $W_0 = 100$, and $p = 1/4$, what is the smallest price at which the poor fellow would be willing to sell this lottery ticket.

$$\left(\frac{1}{4}(200)^{\gamma} + \frac{3}{4}(100)^{\gamma}\right)^{1/\gamma} - 100.$$

D) Suppose that a richer guy has initial wealth of 1500 ducats and the same preferences as the poor fellow, if $X = 100$ and $p = 1/4$, what is the most that this richer guy would offer for this lottery ticket?

The amount Y that he would be willing to pay is the value of Y that satisfies the equation

$$\left(\frac{1}{4}(1600-Y)^{\gamma} + \frac{3}{4}(1500-Y)^{\gamma}\right)^{1/\gamma} = 1500.$$

8 Peter Habit consumes two goods, apples and bananas. He will live for two periods. At the beginning of period 1, he has wealth W and he will not receive any more income other than interest on savings. Let a_t denote his consumption of apples in period t and b_t his consumption of bananas in year t. The price of apples is p_a in both periods and the price of bananas is p_b in both periods. At the beginning of period 1, his preferences over lifetime consumption streams are represented by the utility function

$$U(a_1, b_1, a_2, b_2) = \ln a_1 + \ln b_1 + \beta(\ln a_2 + \ln b_2).$$

If Peter spends less than W in period 1, he will receive interest on his savings at the interest rate r.

A) If $\beta = 1/(1+r)$, how much money will he save and how much of each good will he consume in each period?

Given that he can earn interest at the rate r on his savings, Peter's intertemporal budget is

$$p_a a_1 + p_b b_1 + \frac{1}{1+r}(p_a a_2 + p_b b_2) = W$$

When $\beta = \frac{1}{1+r}$, he will choose consumptions

$$a_1 = a_2 = \frac{W}{p_a(2+2\beta)}$$

and

$$b_1 = b_2 = \frac{W}{p_b(2+2\beta)}$$

Therefore his expenditures in period 1 will be

$$\frac{W}{1+\beta}$$

and his savings will be

$$W - \frac{W}{1+\beta} = \frac{\beta W}{1+\beta}.$$

B) Peter discovers that apples are addictive in the sense that the more apples you consume in period 1, the more apples it takes in period 2 to give you the same "apple rush." To simplify calculations, let us assume that $\beta = 1$, $r = 0$, and $p_a = p_b = 1$. Taking the addictive effect of apples into account, his preferences are represented by the utility function

$$U(a_1, b_1, a_2, b_2) = \ln a_1 + \ln b_1 + \ln(a_2 - \lambda a_1) + \ln b_2$$

where $\lambda > 0$. Suppose Peter makes his decisions about saving and consumption so as to maximize this utility subject to his budget constraint. Compare Peter's apple consumption in periods 1 and 2 when $\lambda > 0$ to the case of no addiction, where $\lambda = 0$. Does his total apple consumption increase or decrease as λ increases? Does his savings increase or decrease as λ increases?

We want to maximize

$$U(a_1, a_2, b_1, b_2) = \ln a_1 + \ln b_1 + \ln(a_2 - \lambda a_1) + \ln b_2$$

subject to

$$a_1 + a_2 + b_1 + b_2 = W$$

At a maximum it must be that

$$\frac{\partial U(a_1, a_2, b_1, b_2)}{\partial a_1} = \frac{\partial U(a_1, a_2, b_1, b_2)}{\partial a_2} = \frac{\partial U(a_1, a_2, b_1, b_2)}{\partial b_1} = \frac{\partial U(a_1, a_2, b_1, b_2)}{\partial b_2}$$

The first equality implies that

$$\frac{1}{a_1} - \lambda \frac{1}{a_2 - \lambda a_1} = \frac{1}{a_2 - \lambda a_1}$$

which implies that $a_2 = (1 + 2\lambda)a_1$ and that

$$\frac{\partial U(a_1, a_2, b_1, b_2)}{\partial a_2} = \frac{1}{a_2 - \lambda a_1} = \frac{1}{(1 + \lambda)a_1}$$

The second and third inequalities then imply that

$$b_1 = b_2 = (1 + \lambda)a_1.$$

Then it follows from the budget equation that

$$a_1 + (1 + 2\lambda)a_1 + 2(1 + \lambda a_1) = W$$

and hence

$$a_1 = \frac{1}{4(1 + \lambda)} W,$$

$$a_2 = \frac{1 + 2\lambda}{4(1 + \lambda)} W$$

$$b_1 = b_2 = \frac{1}{4} W.$$

Taking derivatives, we see that a_1 is a decreasing function of λ and a_2 is an increasing function of λ. while b_1 and b_2 are both constant as λ changes. It follows that period 1 expenditures are decreasing with λ and hence savings increases with λ. Total apple consumption is

$$a_1 + a_2 = \frac{2 + 2\lambda}{4 + 4\lambda} W = W/2$$

and so is constant as λ changes.

C) (For extra credit if time allows:) Construct some alternative models of addiction or habit. For example: What if consuming more apples today makes you like bananas less tomorrow? What if experience with consuming apples in the past makes them taste better in the future? Or suggest any other effect that seems interesting.

I will leave this for you to think about. What is the best way to model various kinds of "addiction"?