Two Properties of Expenditure functions

Proof that $e(p, u)$ is a concave function of p.

Proof: We want to show that for any u and any two price vectors p and p', and for any λ between 0 and 1,

$$\lambda e(p, u) + (1 - \lambda)e(p', u) \leq e(\lambda p + (1 - \lambda)p', u).$$

Let $h = h(p, u)$ and $h' = h(p', u)$, and let $h^{\lambda} = h(\lambda p + (1 - \lambda)p', u)$. We note that $u(h) = u(h') = u(h^{\lambda})$ since $h(p, u)$ is the cheapest consumption vector that yields utility u at price vector p. Then $e(p, u) = ph(p, u) \leq ph^{\lambda}$ (because $u(h^{\lambda}) = u$) Similarly, $e(p', u) = p'h(p', u) \leq p'h^{\lambda}$. It follows from these two inequalities that

$$\begin{aligned} \lambda e(p, u) + (1 - \lambda)e(p', u) &\leq (\lambda p + (1 - \lambda)p')h^{\lambda} \\ &= e(\lambda p + (1 - \lambda p', u). \end{aligned}$$

Notes on Proving Shepherd's lemma.

$$\frac{\partial e(p, u)}{\partial p_i} = x_i(p, u).$$

Proof: $e(p, u) = \sum_j p_j x_j^h(p, u)$. Differentiate this to find that

$$\frac{\partial e(p, u)}{\partial p_i} = x_i(p, u) + \sum_j p_j \frac{\partial x_j(p, u)}{\partial p_i}.$$

Note also that $u(x^h(p, u)) = u$ for all p. Differentiate this

$$\sum_j u_j(x^h(p, u)) \frac{\partial x^h(p, u)}{\partial p_i} = 0.$$

But $u_j(x^h(p, u)) = \lambda p_j$. Now finish proof by substituting u_j/λ for p_j in the above equation and noticing that all the complicated stuff disappears.