When is a CES function concave?

Consider a constant-elasticity-of-substitution function with constant returns:

\[ f(x_1, \ldots, x_n) = \left( \sum_{i=1}^{n} \lambda_i x_i^\rho \right)^{\frac{1}{\rho}}. \tag{1} \]

This function will be concave if \( \lambda_i \geq 0 \) for all \( i \) and \( \rho \leq 1 \). We could prove this by by showing that the Hessian is negative semi-definite, but let’s try another method.

**Step 1-Show that \( f \) is quasi-concave** Let us first show that the this function is quasi-concave. A function is a quasi-concave function if it is a monotone increasing function of a concave function. So let’s look for a simple concave function “hidden inside” of \( f \).

We note that Then

\[ f(x) = g(x)^{\frac{1}{\rho}} \tag{2} \]

where

\[ g(x) = \sum_{i=1}^{n} \lambda_i x_i^\rho. \tag{3} \]

Suppose that \( 0 < \rho \leq 1 \). Then we see from Equation 2 that \( f(x) \) is a monotone increasing function of \( g(x) \). Now it is easy to verify if \( 0 < \rho \leq 1 \), then \( g(x) \) is a concave function, because its Hessian is simply a diagonal matrix with entries of the form

\[ \lambda_i \rho (\rho - 1) x^{\rho - 2}. \]

When \( 0 < \rho \leq 1 \), these terms are all non-positive. Therefore \( f(x) \) is a monotone increasing function of a concave function \( g(x) \) and hence \( f \) is concave.

Suppose that \( \rho < 0 \). Then it must be that \( f(x) = g(x)^{1/\rho} \) is a monotone decreasing function of \( g(x) \). But if it is a monotone decreasing function of \( g(x) \), it is a monotone increasing function of \( -g(x) \). Now when \( \rho < 0 \), the Hessian matrix of \( -g(x) \) is a diagonal matrix with entries of the form

\[ -\lambda_i \rho (\rho - 1) x^{\rho - 2}. \]

When \( \rho < 0 \), these terms are all non-positive and hence \( -g \) is a concave function. Then \( f(x) \) is a monotone increasing function of \( -g(x) \), it must be that \( f \) is quasi-concave.
Step 2-Show that \( f \) is concave

We know that \( f \) is quasi-concave, but a quasi-concave function that is homogeneous of degree 1 must be concave. You will find a proof of this proposition in the notes on “useful properties of quasi-concave and homogeneous functions” appearing in week 5.

Step 3- Generalize to CES functions that are homogeneous of degree less than 1

Where \( f(x) \) is the constant returns to scale function defined in Equation 1, the CES functions that are of degree \( k \) less than 1 take the form \( f(x)^k \) where \( 0 < k < 1 \). Now if we have a concave function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), and an increasing concave function \( g : \mathbb{R} \rightarrow \mathbb{R} \), then if we define the function \( h : \mathbb{R}^b \rightarrow \mathbb{R} \) so that \( h(x) = g(f(x)) \), then \( h \) must be a concave function. (This has an easy proof that you should be able to supply.)