1) Consider the following strategic form game in which Player 1 chooses the row and Player 2 chooses the column. Both players know that this is the payoff matrix and each knows that the other knows this.

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Center</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>1,2</td>
<td>2,1</td>
<td>1,0</td>
</tr>
<tr>
<td>Middle</td>
<td>0,5</td>
<td>1,2</td>
<td>7,4</td>
</tr>
<tr>
<td>Down</td>
<td>-1,1</td>
<td>3,0</td>
<td>5,2</td>
</tr>
</tbody>
</table>

A) Suppose that the two players move simultaneously. Assume that it is common knowledge to both players that neither player will ever play a strictly dominated strategy. What pure strategy outcome (outcomes) would you expect to see? Explain

Discussion of Answer: If it is common knowledge that neither player will ever play a strictly dominated strategy, then since Center is strictly dominated by Left for Player 2, we know that Player 2 will never play Center. Given that Player 2 will never play center, the strategy Down is strictly dominated by Middle. So Player 1, knowing that 2 will not play Center, will never play Down. But if Player 1 will never play Down, then Right is strictly dominated by Left for Player 2. So Player 1 can be assured that Player 2 will not play either Center or Right. But given that Player 2 will play Left, Player 1 is better off playing Up than Middle. the strategy Up for Player 1 and Left for Player 2 would be the outcome.

B) Suppose that in the game with payoff matrix described above, Player 1 moves first and Player 2 moves after having observed Player 1’s move. List the possible strategies for Player 1. List the possible strategies for Player 2. Find the strategy profile that constitutes a subgame perfect Nash equilibrium.

Discussion of Answer: Player 1 moves first and has only three possible strategies, Up, Middle, and Down. Player 2 moves second after observing Player 1’s move. Therefore Player 2’s strategies can depend on Player 1’s move. A single strategy specifies what Player 2 will do in response to each possible action by Player 1. That is to say, a strategy for player 2 must state what action player 2 will take at each information set at which it is player 2’s turn. Since Player 2 has three information sets and three possible actions at each information set, this makes $3 \times 3 \times 3 = 27$ possible strategies. One example of a strategy for Player 2 is ”Go Left no matter what Player 1 does”. Another is ”Go Left if player 1 goes Up. Go Left if player 1 goes Middle and Go Center if player 1 goes
Down.” There are 27 possible such strategies. One could denote each such strategy in the form x/y/z where x is Player 2’s action if Player 1 chooses Up, y is Player 2’s action if Player 1 chooses Middle and z is Player 2’s action if Player 1 chooses Down.

In the only subgame perfect Nash equilibrium, Player 1 chooses Down and Player 2 chooses Right, which gives Player 1 a payoff of 5 and Player 2 a payoff of 2.

2) Lucy asked Charlie to play the following game. “Let’s show pennies to each other. We can either show heads or tails. We will move simultaneously. If we both show heads, I will pay you $3. If we both show tails, I will pay you $1. If one of us shows heads and the other shows tails, then you will pay me $2.”

A) If Charlie agrees to play and if Lucy knows that he is equally likely to play heads or tails, what will Lucy do? Then what will Charlie’s expected profit or loss be?

Discussion of Answer: If Charlie is equally likely to play heads or tails, Lucy’s best response is to play tails for certain. When she does this, Charlie’s expected payoff is $-1/2.

B) Find a mixed strategy Nash equilibrium for the game that Lucy proposed. What strategies does each use? What will be Charlie’s expected profit or loss in this equilibrium? Explain your answer.


3) Alice and Bob are planning to take a trip together. They must choose one of three possible destinations: the mountains, the ocean, or the big city. Alice likes the mountains best, the ocean second, and the city third. Bob likes the ocean best, the mountains second, and the city third. They decide to select their destination by alternately vetoing possibilities until only one option remains. First Alice vetoes a destination. Then, knowing which destination Alice vetoed, Bob vetoes one of the destinations not vetoed by Alice. They will go to the destination that neither vetoed.

A) List the possible strategies for Alice? List the possible strategies for Bob.

Discussion of Answer: Possible strategies for Alice are Veto Mountains, Veto Ocean, Veto City. Bob who moves second has 6 possible strategies. At each of the three information sets at which it is Bob’s turn, Bob needs to choose one of two actions. One possible strategy is “Veto ocean if Alice vetoes mountains, Veto City if Alice vetoes Ocean, and veto ocean if Alice vetoes City.” There are 8 = 2 × 2 × 2 possible such strategies that take the form “veto x if Alice vetoes mountains, veto y if Alice vetoes ocean, and veto z if Alice vetoes city,” where x can be either ocean or city, y can be either mountains or city and z can be either mountains or ocean.
B) Draw an extensive form representation of this game and assign payoffs that are consistent with the story.

C) Show this game in strategic form. Find all of the Nash equilibria. Are all of the Nash equilibria subgame perfect? Explain.

Let us use $x/y/z$ to denote a strategy of the form “veto $x$ if Alice vetoes mountains, veto $y$ if Alice vetoes ocean, and veto $z$ if Alice vetoes city.”

Let the payoffs to either player be 3 for going to one’s favorite place, 2 for going to one’s second favorite place, and 1 for going to one’s third favorite place. Here Alice chooses the row and Bob chooses the column.

**Discussion of Answer:**

<table>
<thead>
<tr>
<th></th>
<th>O/C/O</th>
<th>O/C/M</th>
<th>O/M/O</th>
<th>O/M/M</th>
<th>C/C/O</th>
<th>C/C/M</th>
<th>C/M/O</th>
<th>C/M/M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Veto M</td>
<td>1,1</td>
<td>1,1</td>
<td>1,1</td>
<td>1,1</td>
<td>2,3</td>
<td>2,3</td>
<td>2,3</td>
<td>2,3†</td>
</tr>
<tr>
<td>Veto O</td>
<td>3,2*</td>
<td>3,2*</td>
<td>1,1</td>
<td>1,1</td>
<td>3,2*</td>
<td>3,2*</td>
<td>1,1</td>
<td>1,1</td>
</tr>
<tr>
<td>Veto C</td>
<td>3,2</td>
<td>2,3</td>
<td>3,2</td>
<td>2,3†</td>
<td>3,2</td>
<td>3,2</td>
<td>2,3</td>
<td>2,3†</td>
</tr>
</tbody>
</table>

There are 7 distinct Nash equilibria. In four of these, Alice Vetoes the ocean and Bob vetoes the city when it his turn. These are marked with a *. In these four Nash equilibria, Alice gets to go to her favorite place, the mountains. She gets a payoff of 3 and Bob gets 2. These four N.E. are distinct from each other because each involves Bob taking different actions in the cases where Alice vetoed something else.

There are also two Nash equilibria in which they wind up going to Bob’s favorite place, the ocean. These are marked with †.

A Nash equilibrium will be subgame perfect only if in every subgame where it is Bob’s turn, he takes an action that maximizes his payoff in that subgame. This means that his strategy must to a veto of the mountains by a veto of the city, he must respond to a veto of the ocean by a veto of the city and he must respond to a veto of the city by a veto of the mountains. Therefore the only strategy that Bob can be using in a subgame perfect Nash equilibrium is C/C/M. Therefore the only subgame perfect Nash equilibrium is that in which Alice vetoes the ocean and Bob’s strategy is “veto the city if Alice vetoes either the ocean or the mountains and veto the mountains if Alice vetoes the city.”

The most interesting Nash equilibria that are not subgame perfect are the two marked with daggers. In this case, Bob gets his way because Alice believes that if she doesn’t veto the city, Bob will veto the mountains. But this is not a “credible threat” on Bob’s part. If Alice actually vetoes vetoes the ocean, Bob’s best response is to veto the city and go to the ocean.

4) (Simplified Poker) Archie and Beth each put a dollar into the “pot”. Archie draws a card from a deck of cards. The card is equally likely to be high or low. Archie can see his card, but Beth can not. Archie can either “show” or “raise”. If Archie plays “show”, he shows his card to Beth. If Archie’s card is high, he gets all of the money in the pot and the game ends. If Archie’s card is low, Beth
gets all of the money in the pot and the game ends. If Archie plays “raise”, he puts another dollar into the pot and Beth must choose whether to “pass” or “meet”. If Beth passes, Archie gets all of the money in the pot. If Beth meets, she adds a dollar to the pot and Archie then shows Beth his card. If the card is high, Archie gets all the money in the pot. If the card is low, Beth gets all the money in the pot.

A) Draw an extensive form representation of the game played by Archie and Beth.

B) List the possible strategies for Archie. List the possible strategies for Beth.

Discussion of Answer: Archie sees which kind of card he drew before he decides whether to show or raise. Therefore he has two information sets and four possible pure strategies. These are: (i) Show, whether the card is high or low. (ii) Show if the card is high, raise if the card is low. (iii) Show if the card is low, raise if the card is high (iv) Raise, whether the card is high or low. Let us denote these strategies by SS, SR, RS, and RR, respectively. Beth has only one information set and two strategies. If she is asked for a decision, she can choose either to Pass or to Meet.

C) Describe this game in strategic form. (Payoffs will be the difference between the amount of money one takes out of the pot and the amount one puts in.)

Discussion of Answer: To write a strategic form representation of the game, we find the expected net payoffs to each player for each combination of a strategy by Archie and a strategy by Beth. For example if Archie always plays show, then regardless of Beth’s strategy, Archie will get a payoff of 1 and Beth will get -1 when he draws a high card and Archie will get -1 and Beth will get 1 when he draws a low card. Archie’s expected payoff will be $1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$. Likewise, Beth’s expected payoff will be zero. If Archie plays show when he has a high card and raise when he has a low card and Beth is certain to pass, then Archie will always get +1 and Beth will always get -1. If Archie plays show when he has a high card and raise when he has a low card and if Beth is certain to raise, then when Archie draws a high card he will win 1 and Beth will lose 1 and when Archie draws a low card, he will lose 2 and Beth will win 2. Thus the expected payoff to Archie would be $1 \times \frac{1}{2} - 2 \times \frac{1}{2} = -\frac{1}{2}$ and the expected payoff to Beth would be $\frac{1}{2}$. We can fill out the entire table as follows.

<table>
<thead>
<tr>
<th></th>
<th>Pass</th>
<th>Meet</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>SR</td>
<td>1,-1</td>
<td>$-\frac{1}{2}$, $\frac{1}{2}$</td>
</tr>
<tr>
<td>RS</td>
<td>0,0</td>
<td>$\frac{1}{2}$, $-\frac{1}{2}$</td>
</tr>
<tr>
<td>RR</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>
D) How many pure strategy Nash equilibria does this game have? Explain your answer.

Discussion of Answer: We see from the strategic form representation that if Alice uses the pure strategy Pass, then Archie’s best response is to raise whenever he has a low card. If Archie raises whenever he has a low card, Alice’s best response is Meet. So there can’t be a pure strategy N.E. in which Alice plays Pass. If Alice plays the pure strategy Meet, then Archie’s best response is to Raise whenever he has a high card and show whenever he has a low card. But if this is Archie’s strategy, then Beth’s best response is to pass. So there can’t be a pure strategy equilibrium in which Beth meets. Therefore there is no pure strategy Nash equilibrium.

E) Find a mixed strategy equilibrium in which if Archie sees a high card, he is sure to raise, and if he sees a low card will show with some probability and raise with some probability. In this mixed strategy equilibrium, what is the probability that Archie raises? What is the probability that Beth meets?

Discussion of Answer: Let \( p \) be the probability that Beth passes. Then if Archie plays RS, his expected payoff will be \( 0 \times p + (1 - p) \times \frac{1}{2} = (1 - p)/2 \). If Archie plays RR, his expected payoff will be \( p \times 1 + (1 - p) \times 0 = p \). Archie will be indifferent between these two strategies if \( p = (1 - p)/2 \), which is the case if \( p = 1/3 \). In this case, Archie’s expected utility from either of the two strategies RR and RS is also equal to 1/3. We also note that if \( p = 1/3 \), Archie’s expected utility of playing SS is 0 and his expected payoff of playing SR is \( 1 \times \frac{1}{3} - \frac{1}{2} \times \frac{2}{3} = 0 \). Therefore if Beth plays pass with probability 1/3, Archie will be willing to use a mixed strategy in which he plays both RR and RS with positive probability. Let \( q \) be the probability that Archie plays RS and \( 1 - q \) the probability that he plays RR. Beth will be willing to use a mixed strategy if her expected payoff from playing pass is the same as that from playing meet. Beth’s expected payoff from playing pass is \( q \times 0 + (1 - q) \times 1 = q - 1 \) and her expected payoff from playing meet is \( q \times -\frac{1}{2} + (1 - q) \times 0 = -\frac{q}{2} \). Her payoffs from the two strategies will be equal if \( q - 1 = -\frac{q}{2} \), which is equivalent to \( q = 2/3 \). Therefore there will be a mixed strategy equilibrium in which Beth passes with probability 1/3 and meets with probability 2/3 in which Archie plays the strategy RS with probability 2/3 and RR with probability 1/3.

This means that in the course of play, the probability is 1/2 that Archie draws a high card, in which case he is certain to raise. With probability 1/2 he draws a low card, in which case he will raise with probability 2/3 and show with probability 1/3. So we see that in about \( \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \) of all plays of the game, Archie will “bluff” by raising, even though he has a low card.

5) Consider an infinitely repeated game where the stage game is displayed in the table below, with Player 1 choosing the row and Player 2 choosing the column.

<table>
<thead>
<tr>
<th></th>
<th>Strategy A</th>
<th>Strategy B</th>
<th>Strategy C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy A</td>
<td>6,6</td>
<td>3,8</td>
<td>1,4</td>
</tr>
<tr>
<td>Strategy B</td>
<td>8,3</td>
<td>3,3</td>
<td>0,1</td>
</tr>
<tr>
<td>Strategy C</td>
<td>4,1</td>
<td>1,0</td>
<td>2,2</td>
</tr>
</tbody>
</table>
Consider the symmetric strategy profile where each player plays Strategy A in the first round and continues to choose A, so long as neither player has ever chosen B or C. If B or C is ever chosen, then each player will play strategy C forever after. Find a condition on the discount factor so that this strategy profile is a subgame perfect Nash equilibrium. Show that given this condition, this strategy is in fact a subgame perfect Nash equilibrium.

Discussion of Answer:
If the other player is using this strategy, then a player who plays A every time he gets to play will receive a payoff of 6 in every period. If the discount factor is \( \delta \), the present value of 6 forever is \( \frac{6}{1-\delta} \). If a player were to play strategy B on the first round, he would get a payoff of 8 in the first period and then since thereafter the other player would be playing C forever, the best the player could do in future periods is to get 2 in every period. Getting 8 in the first period and then 2 in all future periods has a present value of \( 8 + \frac{2\delta}{1-\delta} \). So the proposed strategy will be a Nash equilibrium only if

\[
\frac{6}{1-\delta} > 8 + \frac{2\delta}{1-\delta}.
\]

This is equivalent to \( \delta > \frac{1}{3} \).

If \( \delta > 1/3 \), and the other guy is using this trigger strategy, then it doesn’t pay to play anything other than A on the first round. Similar reasoning shows that if nobody has cheated so far, it doesn’t pay to play anything other than A on any round.

To complete the demonstration that this trigger strategy is a Nash equilibrium, we need to ask: Suppose somebody has played something other than A. Is playing C a best response? The answer is Yes. If anybody has ever played anything other than A, then the other guy is going to play C forever. If the other guy is playing C forever, your best response is to play C forever. (This is true because both playing C is a Nash equilibrium for the stage game.)

6) Find all of the Evolutionarily Stable Strategies for the following game, where Player 1 chooses the row and Player 2 chooses the column. Explain your answer.

<table>
<thead>
<tr>
<th></th>
<th>Strategy A</th>
<th>Strategy B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy A</td>
<td>0,0</td>
<td>2,1</td>
</tr>
<tr>
<td>Strategy B</td>
<td>1,2</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Discussion of Answer: In this game, B is the best response to A and A is the best response to B. So the game has no pure strategy symmetric Nash equilibrium. Therefore it won’t have a pure strategy ESS. The remaining candidate is the symmetric mixed strategy Nash equilibrium. There is a symmetric mixed strategy equilibrium for this game in which each plays strategy A with probability \( p \) if

\[
p \times 0 + (1-p) \times 2 = p \times 1 + (1-p) \times 1.
\]
This occurs when \( p = 1/2 \).

Let \( F(p, q) \) be the expected payoff to a player who uses mixed strategy \( p \) when this player is matched against a player using mixed strategy \( q \). We found this mixed strategy by finding \( p \) such that the two pure strategies do equally well against the mixed strategy \( p \). So we need either to check the second order condition, which is \( F(q, p) < F(q, q) \) for \( z \neq p \) or alternatively check the stability of the equilibrium (which necessarily leads to the same answer).

Checking for stability is really easy. Let \( p \) be the fraction of the population that use strategy \( A \). The expected payoff to an individual using strategy \( A \) is then \( 2 - 2p \). The expected payoff to an individual using strategy \( B \) is 1. Strategy \( A \) does better than \( B \) if \( 2 - 2p > 1 \) and worse if \( 2 - 2p < 1 \), or equivalently \( A \) does better when \( p < 1/2 \) and worse when \( p > 1/2 \). The equilibrium is \( p = 1/2 \). Assuming that the type that is getting the higher payoff increases its proportion in the population, the proportions will always move toward equilibrium. That is \( p \) will increase whenever \( p < 1/2 \) and decrease whenever \( p > 1/2 \).

Alternatively, we can check the second order condition for an ESS directly. This is a little more work. We note that for any \( p \),

\[
F(p, q) = 2p(1 - q) + 1 - p.
\]

Therefore

\[
F\left(\frac{1}{2}, q\right) = (1 - q) + 1/2
\]

and

\[
F(q, q) = 2q(1 - q) + 1 - q.
\]

It follows that for all \( q \) between 0 and 1,

\[
F\left(\frac{1}{2}, q\right) - F(q, q) = 1/2 - 2q(1 - q).
\]

Now simple calculus shows that \( 0 \leq 2q(1 - q) \leq 1/2 \) for all \( q \) between 0 and 1 and with strict inequality if \( q \neq 1/2 \). It follows that \( F\left(\frac{1}{2}, q\right) > F(q, q) \) for all mixed strategies with \( q \neq 1/2 \). Therefore the symmetric Nash equilibrium where \( p = 1/2 \) is an ESS.

7) In a symmetric game, a pure strategy is an Evolutionarily Stable Strategy if it is a strict symmetric Nash equilibrium. True or False? If true, explain why this is true. If false, show a counterexample.

**Discussion of Answer:** This is true. A strategy \( x \) is an ESS if in a population where all other individuals use the strategy \( x \), the payoff to a mutant who uses a different strategy is lower than that for normals who use strategy \( x \). Let the payoff to an individual who plays \( x \) in a game with an individual who plays \( y \) be \( F(x, y) \). A strict symmetric Nash equilibrium is a strategy \( x \) such that \( F(x, x) > F(y, x) \) for all \( y \neq x \). In a population made up entirely of \( x \)-strategists, a mutant who plays \( y \neq x \) will almost always be playing with an \( x \)-strategist and hence will get a lower payoff.
An employer has a large workforce, some of whom are self-motivated and some of whom are not. The employer can not tell which workers are self-motivated and which are not, but he knows that the fraction who are self-motivated is $p$. A worker who works hard produces output that is worth $20 to the employer. A worker who loafes produces output worth $5 to the employer. The employer can get a monitor to determine whether a worker is working hard at a cost of $5 per worker.

The employer pays wages of $10 to any worker who is not caught loafing and will pay a wage of $0 to any worker who is caught loafing. Self-motivated workers prefer working hard to loafing, even if the employer doesn’t monitor. A worker who is not self-motivated and gets paid $w$ has a payo of $w$ if he or she works hard and $w + 5$ if he or she loaf.

A) For what range of values of $p$ is there a Bayes-Nash equilibrium in which the employer does not monitor the workers? Explain your answer.

Discussion of Answer:
If the employer does not monitor, then the fraction $p$ will work hard and the fraction $1-p$ will loaf. The firm will pay everyone $10. It will get output worth $20 from those who work hard and output worth $5 from those who loaf. Its expected profit per worker will be $p(20 - 10) + (1 - p)(5 - 10) = 15p - 5$. If the firm monitors, everybody will work and produce output worth $20. The firm will spend $10 on wages for each worker and $5 on monitoring each worker. Its expected profit per worker will be $5. The firm will make more profits by not monitoring if $15p - 5 > 5$, which will be the case of $p > 2/3$.

B) Suppose that $p = 1/3$. Find a mixed strategy Bayes-Nash equilibrium. In this equilibrium, what is the probability that the employer monitors? What is the probability that a worker that is not self-motivated will work hard? What is the probability that a self-motivated worker will work hard?

Discussion of Answer:
The firm will be willing to use a mixed strategy if the fraction of the labor force who work hard is 2/3. We know that the self-motivated workers will all work hard, no matter what mixed strategy the firm uses. But it is possible that a mixed strategy would make those who are not self-motivated indifferent between loafing and working hard and hence willing to use a mixed strategy.

Let $p$ be the probability that the employer monitors. If he works hard, a worker who is not self-motivated will certainly be paid $10 and will have utility $10 - 2 = 8$. If this worker loafes, then with probability $p$ he will be caught and have 0 wages and hence utility of 0 + 5 = 5. With probability $1 - p$ he won’t be caught and will have utility $10 + 5 = 15$. Therefore if the probability of monitoring is $p$, the expected utility who loaf is $5p + 15(1 - p) = 15 - 10p$. A non-self-motivated worker will be indifferent between working and loafing and hence willing to use a mixed strategy if $8 = 15 - 10p$ which means $p = 7/10$.

In the equilibrium, we also need to have the employer indifferent between monitoring or not. This happens when probability that a worker works hard is 2/3. We have assumed that 1/3 of the workers are self-motivated. These
will all work hard. If the fraction \( q \) of the non-self-motivated workers work hard, then the fraction of all workers who work hard will be \( 1/3 + (2/3)q \). We need this fraction to equal \( 2/3 \). This will be the case if \( q = 1/2 \). Thus the mixed strategy equilibrium has the employer monitoring with probability \( 7/10 \), the self-motivated workers always working hard, and the non self-motivated workers, randomizing with equal probability of working hard or loafing.

9) Suppose that members of a large population are randomly matched to play a game of repeated prisoners’ dilemma. In the stage game, if both cooperate, they each get a payoff of 1. If both defect, they each get a payoff of 0. If one cooperates and the other defects, the one who cooperated gets a payoff of \(-1\) and the one who defect gets a payoff of \(T\). After each round of play, each gets to observe how they other played. A fair coin is then tossed. If the coin comes up heads, they will play again. If the coin comes up tails, the game stops. The game continues to be repeated until the first time that the coin comes up tails.

A) Suppose that the population consists almost entirely of individuals who use the following tit-for-tat strategy, *Cooperate on the first round of play. In all later rounds, use the same strategy that the other player used on the previous round.* What is the expected payoff to a tit-for-tat player who is matched with another tit-for-tat player?

**Discussion of Answer:** A tit-for-tat player matched against another tit-for-tat player will get a payoff of 1 for every round that is played. The probability that a \( t \)th round gets paid is \((1/2)^t\). The expected payoff is therefore

\[
2 = \frac{1}{1 - \frac{1}{2}}.
\]

B) What is the expected payoff of a player who plays *Defect, no matter what the other player does* if matched against a tit-for-tat player?

**Discussion of Answer:** This player will get \( T \) on the first round and 0 ever after. The expected value is just \( T \).

C) What is the expected payoff to a player who plays *Defect on odd-numbered rounds, cooperate on even numbered rounds* if matched against a tit-for-tat player?

**Discussion of Answer:** This player would get a payoff of \( T \) in all odd numbered rounds and \(-1\) on even numbered rounds. The expected value of all the \( T \)'s is

\[
T \left(1 + \frac{1}{4} + \frac{1^2}{4} + \frac{1^3}{4} + \ldots\right) = T \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}T
\]

and the expected value of all the -1's is

\[
-1 \times \frac{1}{2} \times \left(1 + \frac{1}{4} + \frac{1^2}{4} + \frac{1^3}{4} + \ldots\right) = -\frac{1}{2} \cdot \frac{4}{3} = -\frac{2}{3}.
\]
So the expected payoff to this strategy is
\[
\frac{4T}{3} - \frac{2}{3} = \frac{2}{3}(2T - 1).
\]

D) For what values of \(T > 1\) would it be true that in a population of tit-for-tat players, rare mutants of types the types mentioned in parts B and C would get lower expected payoffs than tit-for-tat players?

**Discussion of Answer:** We would need \(T < 2\) and \(2/3(2T - 1) < 2\), which also requires \(T < 2\).

E) Suppose in this large population, matching is not entirely random, but with probability 1/2 an individual will be matched with someone who plays the same strategy as oneself and with probability 1/2, one will be matched with a random selection of the remaining population. For what values of \(T\) would it be true that if the population is made up almost entirely of individuals playing *always cooperate* the always cooperate players would get higher expected payoffs than “mutants” who play *always defect*?

**Discussion of Answer:** The always cooperate players would always wind up playing always cooperate types and would get payoffs of 1 so long as the game is played. So their expected payoffs would be \(\frac{1}{1-\left(\frac{1}{2}\right)} = 2\). The always defect types, would with probability 1/2 run into another defector, in which case, they would get a payoff of zero. With probability 1/2, they would run into an always cooperate type, in which case, they would get \(T\) so long as the game is played. The expected value of \(T\) so long as the game is played is \(2T\). So the expected payoff to an always defect type would be \(\frac{1}{2} \times 0 + \frac{1}{2} \times 2T = T\). Therefore the always cooperate types would do better than the always defect types if \(T < 2\) and the always defect types would be able to invade the population only if \(T > 2\).

10) Alice and Bob have differing tastes in movies, but like to be together. If they both go to movie A, Alice will get a payoff of 3 and Bob will get a payoff of 2. If they both go to Movie B, Bob will get a payoff of 3 and Alice will get a payoff of 2. If Alice goes to A and Bob to B, Alice and Bob will each get a payoff of 1. If Alice goes to B and Bob goes to A, they will each get a payoff of 0.

A) Find a symmetric mixed strategy equilibrium for this game. What is the probability that they arrive at the same movie.

**Discussion of Answer:** There is a mixed strategy equilibrium in which Bob goes to movie A with probability 1/4 and to movie B with probability 3/4, while Alice goes to movie A with probability 3/4 and to movie B with probability 1/4. The probability that they will go to the same movie is 3/8. The expected payoff to each of them is 3/2.
B) Suppose that Alice and Bob have not settled on which movie to go to and will not get a chance to discuss it at length. Each can leave a single text message for the other, saying to which movie he or she plans to go. They must choose their messages simultaneously, without knowing the other’s message. If the two messages both name the same movie, they will both go to that movie. If the two messages name different movies, they will both ignore the messages and play an equilibrium symmetric mixed strategy to the original game. Find a symmetric mixed strategy Nash equilibrium to the game in which Alice and Bob each choose a message to send to the other.

Discussion of Answer: If they both send message A, they will go to movie A and Alice’s payoff will be 3 and Bob’s will be 2. If they both send message B, they go to B and Alice’s payoff is 2 and Bob’s is 3. If they send opposite messages, they ignore these messages and use their mixed strategy equilibrium strategies that you found in the previous section. In this case, the expected payoff to each of them is 3/2. So we calculate a mixed strategy Nash equilibrium for a game with these payoffs. Let $p$ be the probability that Bob sends Message A. Alice’s expected payoff if she sends message A is $3p + \frac{3}{2}(1-p)$ and her expected message if she sends message B is $\frac{3}{2}p + 2(1-p)$. Therefore in a mixed strategy equilibrium it must be that $3p + \frac{3}{2}(1-p) = \frac{3}{2}p + 2(1-p)$, which implies that $p = 1/4$. Similar reasoning shows that Alice will send message B with probability 1/4 and message A with probability 3/4.

The probability that they coordinate on the same movie by sending the same message as each other is $\frac{3}{8}$. If they don’t send the same message, then the chances that they meet at the same movie in the subsequent mixed strategy play is $\frac{3}{8}$. So the probability that they meet at the same movie is $\frac{3}{8} + \frac{3}{8} \times \frac{3}{8} = \frac{39}{64} \approx .61$.

C) Suppose that the situation is as in Part B, except that if the first messages between Alice and Bob name different movies, they each get one more chance to send messages about their intentions. Find a symmetric mixed strategy Nash equilibrium to this game and find the probability that they will both go to the same movie.

Discussion of Answer: If the first messages of Alice and Bob are both A, then they go to A and Alice gets 3 and Bob gets 2. If the first messages of Alice and Bob are both B, they both go to B and Alice gets 2 and Bob gets 3. If the two messages differ, they will play the game described in Part B. We found that in that game, the expected payoff to each player is $\frac{15}{8}$. So in a mixed strategy Nash equilibrium for the entire game, their plays on the first message will be a mixed strategy Nash equilibrium for the game with the following payoff matrix, where Alice chooses the row message and Bob chooses the column message.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3,2</td>
<td>15/8, 15/8</td>
</tr>
<tr>
<td>B</td>
<td>15/8, 15/8</td>
<td>2,3</td>
</tr>
</tbody>
</table>

If Bob sends message A with probability $p$, Alice’s expected payoff from message A will be $3p + (1-p)\frac{15}{8}$ and her expected payoff from message B will
be \( \frac{15}{10} p + 2(1 - p) \). These two payoffs are equal when \( p = \frac{1}{10} \). So on the first message, Bob will say movie A with probability \( \frac{1}{10} \) and movie B with probability \( \frac{9}{10} \). By symmetry, on the first message, we will find that Alice will say movie A with probability \( \frac{9}{10} \) and B with probability \( \frac{1}{10} \). On the second message, Bob says movie A with probability \( \frac{1}{4} \) and B with probability \( \frac{3}{4} \) and Alice says B with probability \( \frac{1}{4} \) and A with probability \( \frac{1}{4} \).

The probability that Alice and Bob will coordinate on the same movie in their first message is \( 2 \times \frac{1}{10} \times \frac{9}{10} = \frac{18}{100} \). If they don’t coordinate on the first message, they are left with the game described in Part B. There we found that the probability they would get to the same movie was \( \frac{39}{64} \). Therefore when they get to send two messages, the probability that they will get to the same movie is

\[
\frac{18}{100} + \left( \frac{82}{100} \times \frac{39}{64} \right) \approx .680.
\]