Lecture B-1: The Economic Allocation Mechanisms: An Introduction

Warning: These lecture notes are in a very preliminary state and contain a large number of mistakes!

1. A basic setup

Let us focus on the following simple question: how do a group of individuals who face a shortage of resources allocate those resources among themselves?

In the model we deal with here, a society which consists of a set of agents $I = \{1, \ldots, n\}$ and a set of houses $H = \{1, \ldots, m\}$. For simplicity assume that $m \geq n$ (the case $m < n$ can be treated similarly).

Each agent can possess (or consume) only one house.

Each agent $i$ has a strict ordering $\succ_i$ over the set $H$.

We are interested in mechanisms which allocate the houses among the agents.

A feasible allocation is a one-to-one function from $I$ into $H$. This reflects the assumption that an agent cannot hold more than one house and that no individual can be without house. Thus, if $x$ is a feasible allocation then $x(i) \neq x(j)$ for all $i$ and $j$.

We say that an allocation $x$ is efficient if there is no other allocation $y$ such that for all $i$ we have $y(i) \succeq_i x(i)$ and for at least one $j$, $y(j) \succ_j x(j)$.

2. Mechanisms:

(0) Big Brother: A super computer calculates a number which describes $v_i(h)$ which is the value the super computer attaches to $i$ holding $h$. The computer then looks for an allocation which maximizes the sum $\sum_i v_i(a(i))$ over all allocations. In this process the super computer might but not necessarily take into account the individuals’ preferences.

(1) Lottery: the houses are allocated to the individuals by a fair lottery. This procedure may seem fair but can yield unfair outcomes. The realized allocation
might be inefficient. Note that this procedure is independent of the individuals’ preferences.

(2) Sequential Dictatorship with Random Ordering: An ordering of the agents is determined by a fair lottery. Then the agents are called one by one to choose a house from those yet unchosen. This procedure is sensitive to the individuals’ preferences. The procedure is fair ex-ante and yields an efficient ex-post outcome (to be shown later).

(3) Borda: The preferences are translated into "points" (the more preferred houses get more points and a computer calculates the allocation which maximizes the sum of the "points" over all individuals.

(4) Determine which house is considered most popular by the largest number of individuals. Allocate it randomly to one of those individuals. Continue the process with the remaining houses and the remaining agents.

(5) The individuals play a game. In each round each individual makes a demand (for a house). If the demands are compatible, the allocation is implemented; if not the game continues to the next round. For the case of $n = 2$, the equilibrium outcome of the game must be an agreement. However, for $n = 3$ this isn’t the case: there are equilibria with no agreement such as when all agents always demand house 1. Then, no one can unilaterally deviate and gain.

(6) Houses are “auctioned” one by one. House 1 is auctioned first. The agents are required to do push ups and the one who does the most will get the house. We will continue with the rest of the agents and the rest of the houses.

3. The jungle

In this section as well as the next one, we will add some more information to the model and talk about mechanisms which rely on that extra information.

Let us start with an unconventional model of allocation – the model of the jungle.

To the model we now add information about the relative power of the individuals. Let $S$ be a power relation on the set of individuals. The statement $iSj$ means that $i$ is stronger than $j$ and can take from him whatever asset he holds. We assume that the relation $S$ is an ordering and without loss of generality assume $1S2S3\ldots Sn$.

We look for an allocation of the houses which will remain “stable” despite the
potential forces operating in the model. We say that an allocation \( a = \{a(i)\}_{i=1} \) is a Jungle Equilibrium if there is no \( i \) such that there exists a house \( h \) which is not held by one of the individuals stronger than him (namely \( h \notin \{a(1),\ldots,a(i-1)\} \)), or in other words, is abandoned without a possessor or is possessed by a weaker individual, such that \( h >_i a(i) \). In other words an allocation is not a jungle equilibrium if \( i \) can improve his situation either by replacing his house with a house held by a weaker agent or one not possessed by anyone.

Claim 1: A jungle equilibrium exists

Proof: Consider the allocation which is obtained by calling the agents one by one, according to the order of power, to pick a house from those not allocated earlier in the process. Namely, agent 1 picks first and chooses his preferred house \( a(1) \). Agent 2 chooses then \( a(2) \) the best house according to his preferences from among \( H - \{a(1)\} \) and so on. The allocation is a jungle equilibrium since each agent \( i \) possesses a house which is the best from \( H - \{a(1),\ldots,a(i-1)\} \).

Claim 2: A jungle equilibrium is efficient.

Proof: Consider a feasible allocation \( b \) such that \( b(j) \succ_j a(j) \) for all \( j \) with at least one strict inequality. Let \( j^* \) be the strongest agent for whom \( b(j) \succ_j a(j) \). It must be that \( b(j^*) \in \{a(1),\ldots,a(j^*-1)\} \) and thus there is \( i \neq j^* \) (that is \( i < j^* \)) for which \( a(i) \neq b(i) \) and since we do not allow indifference it must be that \( b(i) >_i a(i) \) which is a contradiction.

4. Markets

The mechanism of competitive equilibrium is based on the notion of ownership. Each agent \( i \) starts the interaction owing a house \( e(i) \). There are no houses which lack initial owner so we confine ourselves to the case of \( m = n \).

A candidate for a competitive equilibrium is a pair which consists of:

1. a price vector \( (p_h) \): one price for each house \( h \).
2. a vector of actions, \((x^i)\): one action (a purchase of a house) for each agent \( i \).

For a pair consisting of a price vector and an action vector to be a competitive equilibrium we require that:
(1) $x^i$ maximizes $\xi_i$ given the budget set $\{x|p_x \leq p_e(0)\}$. That is, for every $i$ the action $x^i$ is optimal given the budget set. In other words, the house $x^i$ is the best house for agent $i$ from the set of houses he can afford.

(2) $\{x^1, \ldots, x^n\} = H$ That is, all demands can be satisfied, or, in other words, the demand for each house is equal to its supply which is 1.

**Claim 1: A competitive equilibrium exists.**

**Proof:** A “top cycle” is a set of agents $J_1$ such that the set of top priorities for all members of $J_1$ are the houses owned initially by the members in $J_1$. To show that a top cycle exists - start with the house $1$ wants most, denoted by $M_1$, and let $i_1 = 1$. This house, $M_1$, is owned by an agent who will be denoted by $i_2$ (that is $e(i_2) = M_1$). Let $M_2$ be the house that $i_2$ likes most. This house is owned by an agent called $i_3$. This construction eventually must lead to a member $i_{T+1}$ who appeared in the list earlier as $i_i$. The set $J_1 = \{i_1, \ldots, i_T\}$ is a top cycle. Assign to each $i$ in $J_1$ the house he most prefers.

Continue with the rest of the agents $(N-J_1)$ and the rest of the houses (those which are not owned initially by the members of $J_1$). In this way we will partition $N$ into disjoint sets $J_1, J_2, \ldots, J_L$. Attach prices $p_k$ to all houses owned initially by the members of $J_k$ so that $p_1 > p_2 > \ldots > p_L > 0$. Verify that this price vector and the assignment of houses is a competitive equilibrium.

**Discussion:** An agent’s “power” in the market is the result of the complementarity of the preferences held by agents. Power in the market is affected by the initial holdings but is not independent of the preferences.

**Claim 2: A competitive equilibrium allocation is efficient.**

**Proof:** Let the price vector $(p_1, \ldots, p_n)$ and the allocation $(z^1, \ldots, z^n)$ be a competitive equilibrium. Assume that the allocation is not efficient. Thus, there is an allocation $(y^1, \ldots, y^n)$ such that some agents are better off and none is worse off. In other words, for all $i$, $y^i \succeq_i z^i$ and for at least one agent the inequality is strict. Denote by $J$ the non empty set of agents for whom $y^i \neq z^i$. Since we do not allow indifferences $j$, $y^j \succ_j z^j$ for all $j \in J$ and thus it must be that $p_{z^j} > p_{y^j}$ for all $j \in J$ and
for all \( j \notin J \). Thus \( \sum p_h = \sum p_{z^j} > \sum p_{y^j} = \sum p_h \), which is a contradiction.

**Problem Set B-1:**

**Problem 1:**
Show that with strong preferences the jungle equilibrium is unique.

**Problem 2:**
Show the existence of jungle equilibrium in a world with \( K \) commodities where an initial bundle \( w \) can be divided in any way between the \( n \) agents. Assume that each agent has a classical preference relation (satisfying continuity, strict monotonicity and strong convexity) over the set of bundles and is restricted to consuming a bundle within a set of bundles \( X^i \) which satisfies compactness, free disposal and convexity. Show that a jungle equilibrium in this case is unique and efficient.

**Problem 3:**
Show that if each agent has preferences with indifferences there might be a jungle equilibrium which is not efficient.
Show that if each agent has preferences with indifferences there might be a competitive equilibrium which is not efficient.

**Problem 4:**
A second fundamental welfare theorem: Let \((a(i))_{i=1,...,n}\) be an efficient allocation. Show that the allocation is also the outcome of a competitive equilibrium given some initial allocation of the houses.

**Problem 5:**
Let \((a(i))_{i=1,...,n}\) be a jungle equilibrium allocation. Let \( w^i = a(i) \). Show that there are competitive equilibrium prices for this market with no trade taking place.

**Problem 6:**
Can a competitive equilibrium necessarily be obtained by a chain of pairwise exchanges where each exchange has the property that the exchange improves the situation of the two parties?

**Problem 7:**

Construct and analyze a model which will be similar to the one analyzed above with $n + 1$ agents and $n$ houses. The “new” agent 0 initially owns the $n$ houses and cares only about an additional good, “money”. Each agent $i$ ($i = 1, \ldots, n$) initially holds $m_i$ and care lexicographically first about the house he will own and secondary about the money left in his pocket after purchasing the house.