1. Travelers from City A to City B must cross two rivers to make the journey. There are no bridges, but each river has a ferry boat. Everybody who travels from A to B makes a round trip. The two ferries are owned by two different boatmen, each of whom sets a price for crossing the river. Each boatman has a marginal cost of $10 per passenger for taking a passenger across the river and back again. Each sets his own price for a round trip ticket in an effort to maximize his own profits. The total cost to any traveller of going from City A to City B and back is $C = 15 + P_1 + P_2$ where $P_1$ and $P_2$ are the prices charged by ferryboat 1 and ferryboat 2, respectively. The number of travellers per day who choose to make the trip is $100 - C$ where $C$ is the cost of making the trip.

A) If the two boatmen set their price independently and each believes that the other's price will be the same as his own, what price should each charge in order to maximize his own profits? How many people per day will make the trip?

**Answer:**

Boatman 1’s profits are

\[ P_1(100 - 15 - P_1 - P_2) - 10(100 - 15 - P_1 - P_2) = (P_1 - 10)(85 - P_1 - P_2) = 95P_1 - P_1^2 - P_1P_2 - 850. \]

Set the derivative of this expression with respect to $P_1$ equal to 0, which gives you

\[ 95 - 2P_1 - P_2 = 0. \]

Boatman 2’s profits are

\[ (P_2 - 10)(85 - P_1 - P_2). \]

Set the derivative of this expression with respect to $P_2$ equal to zero and you have

\[ 95 - 2P_2 - P_1 = 0. \]

If both boatmen are maximizing their profits, given the other guy’s price, these two equations must both be satisfied. Solving these equations we have $P_1 = P_2 = 95/3$. Then the number of people who make the trip is

\[ 100 - 15 - P_1 - P_2 = 85 - 63\frac{1}{3} = 21\frac{2}{3}. \]

Remark: This problem is pretty simple if you simply write profits as a function of your own price and find the price that maximizes
these profits. Several people tried to use the “marginal revenue equals marginal cost” rule on this problem. This rule will give the right answer if you correctly calculate marginal revenue. Marginal revenue is the derivative of revenue with respect to quantity sold, not the derivative with respect to price. Setting the derivative of revenue with respect to price equal to marginal cost gives you a wrong answer. When you increase quantity by one unit, you increase costs by marginal cost. But when you increase price by one unit, you reduce output and hence reduce total cost. For the demand function in this problem, when you increase price by one unit, you reduce quantity by one unit. Therefore if you are maximizing profits, you want the derivative of revenue with respect to price to be minus the marginal cost.

B) If a single company owns both ferries and sets a single price for using both ferries in such a way as to maximize total profits, what price will that be and how many people per day will make the trip?

Answer:
Since the monopolist has to operate both boats, the marginal cost for a trip will be 20. At price $P$, the monopolist’s profits are

$$(P - 20)(100 - 15 - P) = 105P - P^2 - 1700.$$  

The derivative with respect to $P$ is zero when

$$105 = 2P$$

and hence $P = 52.5$. At this price, the number of people who make the trip is 32.5.

I gave full credit to those who solved the problem for a marginal cost of 10 and extra credit to those who made marginal cost 20. With a marginal cost of 10, the answer is $P = 47.5$.

C) Suppose that each boatman can set his own price, but boatman 2 promises that he will charge exactly the same price that boatman 1 chooses. If boatman 1 believes boatman 2’s promise, what price should he set to maximize his own profits?

Answer: In this case, each boatman charges half the monopoly price, so that the total cost for a passenger is the same as when a monopoly charges a single price for the whole journey. To show that this is true, note that if boatman 1 charges an amount $P_1$, boatman 2 will also charge $P_1$, so that the cost to users will be $15 + 2P_1$ and the number of passengers is $(100 - 15 - 2P_1)$. Profits of boatman 1
will then be \((P_1 - 10)(100 - 15 - 2P_1) = 105P_1 - 2P_1^2 - 850\). Setting the derivative of profits equal to 0, we find that \(P_1 = 26.25\). Then \(P_2 = P_1\) and so \(P_1 + P_2 = 52.5\), which is the same total price as that charged by a monopolist. (NOTE: I had a calculation error in the answer when I first posted it. I think this is now correct. 6/4/10)

2. An aquifer is an underground layer of water-bearing permeable rock from which groundwater can be extracted using a water wells. The amount of water contained in any aquifer is finite, though replenished gradually by rainfall. This question concerns an imaginary aquifer which we will call the Pronghorn. The Pronghorn aquifer lies beneath an area of several thousand acres of tillable land. Several hundred farmers own land above the aquifer and they have water rights which allow them to drill as many water wells as they wish. As more wells are drilled, the stock of water in the aquifer diminishes and as a result the amount of water that can be pumped from each well also diminishes.

Let \(S\) be the total amount of water in the aquifer. Let \(F\) be the total amount of water that is pumped out of the aquifer per year. Some, but not all of the water that is pumped out is replaced by rainfall. The amount of water contained in the aquifer is given by \(S = 1,000 - F/2\). The total amount of water that is pumped out per year depends on the number of wells drilled and on the amount of water in the aquifer according to the relation \(F = .01SW\).

A) If the number of wells is \(W\), solve for the amount of water \(S\) in the reservoir. Also, solve for the amount of water \(F/W\) pumped out of each well per year as a function of \(W\).

Answer:

\[ S = \frac{1000}{1 + .005W} \]

\[ F/W = \frac{10}{1 + .005W}. \]

B) Suppose that the annualized cost of building and maintaining a well is $10,000 and that the value per unit of water pumped out is $4,000. If individuals drill wells so long as it is profitable to do so, what is the equilibrium number of wells drilled?

Answer: Solve the equation

\[ 4000 \frac{F}{W} = 10,000 \]
This implies that
\[
\frac{40,000}{1 + .005W} = 10,000.
\]

The solution is \( W = 600 \).

C) Suppose that a single owner controlled drilling into the aquifer, write an expression for his total annual profits as a function of the number of wells that are drilled. How many wells should he allow to be drilled in order to maximize his profits?

**Answer:** Total profits would be
\[
4000 \frac{10W}{1 + .005W} - 10,000W.
\]

The derivative of total profits with respect to \( W \) is
\[
\frac{40,000}{(1 + .005W)^2} - 10,000.
\]

The solution is \( W = 200 \).