Is It One Break or Ongoing Permanent Shocks 
That Explains U.S. Real GDP?

A Bayesian Analysis Using an Unobserved Component Model*

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Abstract

The relative importance of permanent (trend) versus cyclical shocks to GDP has been a central issue in macroeconomics since the work of Nelson and Plosser (1982). Morley et al. (2003) find large trend shocks. In contrast, Perron and Wada (2009) argue that allowing for a onetime change in the mean growth rate at 1973:1 nearly eliminates evidence for any trend variance. We re-estimate the Perron and Wada model conditional on a trend break having occurred at any one quarter. We then average the conditional estimates of the trend variance over the probability that the break occurred in a specified quarter. We do this with a Bayesian model average which incorporates break date uncertainty into a trend-cycle decomposition of U.S. real GDP. We find a break date around 2006:1. The median estimate of the size of the drop in the mean growth rate is even larger than the decrease estimated by Perron and Wada. Allowing for a break significantly reduces estimates of trend variance, as Perron and Wada suggest. However, enough spread remains in the posterior distribution of the trend variance to indicate that available data does not definitively settle the question. Although a model with a fixed break date is preferred over an uncertain break date by the Bayes factor, assuming a fixed break date at 2006:1 makes only a modest difference in the posterior for the trend variance.

JEL-Classification: C11, C22, E32
Keywords: trend-cycle decomposition, unobserved component model, structural break, uncertain break date, Bayesian analysis
1 Introduction

The relative importance of a nonstationary component in real output has been a key issue in our understanding of the business cycle at least since Nelson and Plosser (1982). Using an unobserved component model, Morley et al. (2003) (hereafter MNZ) find that the trend component of U.S. real GDP has a unit root and accounts for most of the fluctuations in output. However, Perron and Wada (2009) (hereafter PW) find that once a structural break in the mean growth rate of real GDP—exogenously set as occurring in 1973:1—is incorporated in MNZ’s specification, the nonstationary component essentially disappears. One can think of the dispute as being between a one-time, very large permanent change in the growth rate versus ongoing permanent shocks to the level of real output.

In this paper we take the MNZ/PW model and allow a break in the GDP process to occur at any date, rather than in a pre-specified quarter. Our estimation process takes place within a Bayesian model averaging framework. It may be useful to think of the analysis that follows as taking place in several steps. First, we estimate the probability that a trend break occurred on a specified date for each date in the sample. We then estimate the probability distribution for the model parameters, notably for the trend variance, conditional on a break having occurred on a given date. Finally, we integrate the conditional probability distributions with respect to the probability of the break date to obtain an unconditional distribution. This final step creates
a weighted average for parameter estimates where weights are given by the
estimated (posterior) break date probability.

We estimate our model using the U.S. quarterly real GDP from 1947:1
to 2013:3. We find that the most likely break dates cluster around 2006:1,
so in what follows we use 2006:1 as if it were specified ex ante to be the
break date when we make comparisons with our model which allows for
the break date to be uncertain\footnote{PW restricted their sample to 1947:1 through 1998:2, using the same data as MNZ. Some of the differences between our findings and those in PW and MNZ result from the longer sample period rather than from methodological differences. Most notably we choose 2006:1 as the break date, which is outside the original data period. In Section 4, we comment briefly on results for the shorter sample.}. We estimate small trend variances for the
models incorporating break dates associated with the bulk of the posterior
distribution for the date of the break. As a result, when we integrate the
conditional distributions across break dates the mode of the unconditional
distribution is close to the PW finding of a small trend variance. However,
the probability weight on the larger trend variance is not negligible. The
unconditional distribution is bimodal with the lower mode being fairly close
to the MNZ finding. Or to say it in a different way, the evidence weighs in
the direction of a small trend variance, but the evidence is not strong enough
to be conclusive.

Our primary interest is in the unconditional distribution for trend vari-
ance and the corresponding estimation of the trend and cycles. However, one
might wish to compare directly the PW fixed-break model and the MNZ no-
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model both assume that a break occurred at some point. Perhaps the data prefer the no-break MNZ version. Or, conditional on there being a break, we can ask whether the data clearly identify 2006:1, the most likely break date we estimate, as the correct choice of break. PW conducted extensive robustness checks and concluded that the evidence strongly favors a single break. However, model comparisons can be problematic when one parameter is on the edge of the parameters space (the trend variance equaling zero). Non-standard tests are required to compare different break dates\(^2\).

We avoid the complication of classical tests by implementing Bayesian model comparisons. It is straightforward to compare models in a Bayesian framework by simply comparing the marginal likelihoods of different models. We find that the evidence supports a break and further a fixed break at 2006:1 over the uncertain benchmark. The posterior distributions for the trend variance for the uncertain break date and fixed break date models are quite similar, although the latter has a mode at a slightly lower value than does the former. Both break date models have posteriors that mostly cover lower trend variances than does the posterior for a no-break model. However, the break date models have large enough posterior right tails that we conclude that the data does not speak clearly enough to decisively settle the issue.

As a natural output of our Bayesian approach, the GDP trend and cycle components can be estimated without basing on the point estimates of the

\(^2\)See, for instance, Morley and Eo (2013).
parameters. Given the uncertainty as to the size of nonstationary component, it is valuable to avoid the potential sensitivity caused by conditioning on a single point estimate. Our trend estimate suggests a major and significant growth rate slowdown in the past decade and some permanent loss in GDP during the last recession.

The remainder of our paper is organized as follows. Section 2 specifies our model with an uncertain break date in the mean growth rate. Section 3 presents our Bayesian approach. Section 4 presents the results. Section 5 concludes the paper.

2 The benchmark model: an unobserved component model with an uncertain break date

Both MNZ and PW adopt an unobserved component (UC) model with a random walk trend component, an AR(2) cycle component and correlation between the trend and cycle shocks.

The unobserved component, trend-cycle decomposition model is:
In the above model, $y_t$ is the logarithm of real GDP, which is the sum of the trend component $\tau_t$ and the cyclical component $c_t$. $\tau_t$ follows a random walk with drift $\mu_t$. $\mu_t$ is the mean growth rate of the real output, which may or may not be constant. As PW argue for the possibility of a structural break in the mean growth rate, we also allow for such a structural change here. As shown by (3), there is a permanent change in the mean growth rate $\mu_t$ one period after the break date denoted by $T_b$, with the size of change denoted by $d$. $1(t > T_b)$ is an indicator function that is zero until the break date, and takes the value 1 afterwards. In other words, the mean growth rate equals $\mu$ in the earlier sample periods, and $\mu + d$ after the break date. The cyclical component $c_t$ is assumed to be a stationary AR(2) process. $\eta_t$ and $\epsilon_t$ are the shocks to the trend and cycle respectively. We allow for contemporaneous correlation ($\rho$) between trend and cycle shocks in the model as follows\(^3\):

\[^3\text{In this UC representation with an AR(2) cycle, the correlation between } \eta_t \text{ and } \epsilon_t \text{ is identified if neither } \sigma_{\eta} \text{ nor } \sigma_{\epsilon} \text{ equals zero.}\]
The model given in (1)-(5) nests both the model of MNZ, with $d = 0$ and

\[
\begin{pmatrix}
\eta_t \\
\epsilon_t
\end{pmatrix}
\sim i.i.d. N
\left(\begin{bmatrix}
0 \\
0
\end{bmatrix},
\begin{bmatrix}
\sigma^2_{\eta} & \rho \sigma_{\eta} \sigma_{\epsilon} \\
\rho \sigma_{\eta} \sigma_{\epsilon} & \sigma^2_{\epsilon}
\end{bmatrix}
\right)
\]

(5)

and the PW specification, with $d \neq 0$ and $\sigma_{\eta} \approx 0$. PW find that once we allow for a break in the mean growth rate of the trend component, the stochastic variation for trend GDP becomes insignificant except for periods around the break date. In contrast to the results in MNZ, the standard deviation of the trend shock is estimated to be close to zero. Specifically in the shorter sample used by both PW and MNZ, the standard deviation of the trend shock is estimated to be 1.2368 in MNZ, while the estimate of the same parameter in PW is 0.104 after allowing for the one-time trend break. PW argue that their estimation captures a growth rate slowdown around 1973:1 and a deterministic broken-trend model is a better description for the quarterly U.S. real GDP.

Opinions from the literature vary considerably on the existence and timing of this structural break in the mean growth rate. For instance, Ben-David and Papell (1998) conduct a series of classical hypothesis tests and reject the significance of trend breaks in U.S. real GDP between 1950 and 1990. Chen and Zivot (2010) conduct Bayesian estimation with 130 years of annual data and find that the only possible structural break after 1947 takes place between 1947 and 1952. Perron and Wada (2009) obtain evidence for the structural
trend break occurring around 1973 using the unobserved component model
presented above.

In the succeeding sections, we will let the break date be uncertain and
estimated by the data. We take the model with an uncertain break date as
our benchmark. We offer comparisons to a fixed break in 1973:1 (labeled
PW), as originally specified by PW for their shorter sample; to a fixed break
at the most likely break date, 2006:1 given the full sample period (labeled
PW06); and to the no break date case (labeled MNZ).

3 Bayesian estimation

In this section, we estimate the benchmark model defined by (1)-(5) us-
ing a Bayesian approach. We allow the break date $T_b \in [1, T - 1]$ (i.e.
[1947:1, 2013:2]) to be estimated together with other parameters. Our Bayesian
estimation is conducted using the MCMC Gibbs sampling approach.

The Bayesian approach provides several advantages. Firstly, the break
date and other parameters can be jointly estimated easily, although with
slightly increased computation cost. Secondly, Bayesian approaches provide
direct finite sample inferences. Thirdly, the trend-cycle decomposition can
be obtained as one of the direct outputs of the Bayesian Gibbs sampling,
incorporating the effects of the parameters uncertainty.

Our model can also be estimated approximately by an approximate av-
eraging approach that only utilizes the MLE results. The approximate ap-
proach has potentially larger approximation errors but lower computation
cost. The details of the approximate approach can be found in the working
paper version of this paper.

3.1 Bayesian estimation

The model containing (1)-(5) can be rewritten into state space form:

\[ y_t = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x_t \]  
\[ x_t = \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} 1(t > T_b) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ \epsilon_t \end{bmatrix} \]

where \( x_t = [\tau_t, c_t, c_{t-1}]' \).

In order to ensure that the estimated covariance matrix is positive semidefinite, we decompose the covariance matrix in the following way:

\[
\begin{pmatrix}
\sigma_\eta^2 & \rho\sigma_\eta \sigma_\epsilon \\
\rho\sigma_\eta \sigma_\epsilon & \sigma_\epsilon^2
\end{pmatrix}
= \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}
\]

and directly estimate \( \{\sigma_1, \sigma_2, b\} \) instead of the covariance matrix parameters \( \{\sigma_\eta, \sigma_\epsilon, \rho\} \). The posterior samples for the covariance parameters are obtained
through transformation.

We specify independent proper priors for all parameters estimated. Inverse gamma priors $\text{IG}(100, 0.5)^4$ are assumed for $\sigma_1^2$ and $\sigma_2^2$. These priors are diffuse and do not have finite moments. Therefore, a heavy weight will be put on sample information. We assume somewhat informative normal priors for $\phi_1 + \phi_2 \sim N(0.5, 1)$, $\phi_2 \sim N(-0.5, 1)$ and $b \sim N(0, 1)$. We impose truncations for $\mu \in [0, 2]$ and $d \in [-1, 1]$. We assume uniform priors for $\mu$ and $d$ over the truncated areas and develop truncated normal posteriors accordingly$^5$. While the above priors are broadly consistent with estimates from the literature, we emphasize that our estimation results are robust to more diffuse priors$^6$. Lastly, we assume a flat proper prior for $Tb$ such that all dates from 1947:1 to 2013:2 have equal probability to be the break date in the mean growth rate. Therefore, the joint prior density is the product of all the above marginal prior densities. We present prior moments and quantiles in Table 2.

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$^4$We follow Koop (2003) for the definition of inverse gamma (IG) distribution. If $x > 0$ follows inverse gamma distribution $\text{IG}(s^{-2}, \nu)$, the probability density function of $x$ is defined as:

$$
f(x; s^{-2}, \nu) = \left( \frac{2s^{-2}}{\nu} \right)^{-\nu} \frac{1}{\Gamma(\frac{\nu}{2})} x^{-\frac{\nu}{2}-1} \exp(-\frac{s^{-2}}{2} x)
$$

where $\Gamma(\cdot)$ is the gamma function.

$^5$Such truncation is assumed to avoid unreasonable draws for $\mu$ and $d$. As noted in de Pooter et al. (2008), $\mu$ and $d$ are nearly unidentified when the samples of $\phi_1$ and $\phi_2$ get very close to the non-stationary region. In this case, arbitrary real values for $\mu$ and $d$ can be drawn and cause the Gibbs sampler to have difficulty in moving away from the non-stationary region.

$^6$As robustness check, we set the variance for the normal priors to be 10 and priors for $\sigma_1^2$ and $\sigma_2^2$ to be $\text{IG}(100, 0.1)$, and get similar results.
We use the Gibbs sampling approach to draw posterior samples for parameters including the break date. Sampling the joint posterior distribution of parameters can be conducted by sequential sampling from the conditional distributions. Details on the Gibbs sampling are summarized in Appendix A.

We run our Gibbs sampler for 300,000 times and save every 10th draw to reduce the autocorrelations among samples. We thus obtain 30,000 draws from the Gibbs procedures and discard the first 10,000 to avoid the effect of the initial values. To verify the convergence of the Gibbs sampler, we divide our samples, excluding the burn-in draws, into three sets—a first set of 6000 draws, a middle set of 7000 draws and a last set of 7000 draws. We find that the posterior distribution and estimates based on the three subsets don’t vary much, suggesting the convergence of our MCMC samples.

3.2 Bayesian model comparison

PW’s specification allows for a fixed break date and MNZ’s model assumes that there is no permanent changes in the mean growth rate. Comparing our model with the PW and MNZ specification provides insights on how likely there is a structural break in the mean growth rate and whether we can be certain about the choice of the most likely break date.

We compare two models based on their posterior odds. To be specific, posterior odds of Model$_i$ versus Model$_j$ is defined as
If we assume equal prior probability for two models, the posterior odds can be simplified as the ratio of marginal likelihoods, also known as the Bayes factor \( B_{ij} = \frac{f(Y|M_i)}{f(Y|M_j)} \).

Given that the PW and the MNZ models are both nested in our benchmark model, they can be estimated by the Gibbs approaches described in Section 3.1 but skipping the unnecessary blocks. We use same priors for the unrestricted parameters in the nested models as we do for the benchmark model. The Bayesian marginal likelihood of each model can be numerically computed using Chib (1995)’s method.

For convenience, we will construct twice log Bayes factor \( 2 \log(B_{ij}) \) as the statistic for model comparisons, which is a monotonic transformation of the Bayes factor. If \( 2 \log(B_{ij}) \) is positive (negative), we prefer \( M_i \) (\( M_j \)).

Kass and Raftery (1995) and Raftery (1995) suggest using the following criteria for significance of model comparisons.

As a metric for what follows, a \( 2 \log(B_{ij}) \) of 2 corresponds to model odds of 2.7 to 1 and a \( 2 \log(B_{ij}) \) of 6 corresponds to odds of 20 to 1. While there is not an exact frequentist comparison, note the usual 5 percent significance.
level implies that if the null is true the odds are 19 to 1 against rejecting while a 1 percent significance level implies 99 to 1 odds.

4 Results

We present the Bayesian estimation results for the full sample period in Table 2. Posterior statistics after discarding the initial burn-in samples are shown. 90% highest posterior density regions are also provided in Table 2. The Bayes factor prefers the single break date model to an uncertain break date model, although we caution again that the break date is chosen precisely to maximize the marginal likelihood. The posterior distributions for the trend variance are similar for the two models. The no break date model is strongly rejected. The upper panel in Table 4 provides more details of model comparisons. Note that while the estimated trend variance is notably smaller when a break is allowed for, so is the cyclical variance. On the question of whether variance is dominated by trend shocks or cyclical shocks, the Bayesian estimates come out somewhere in between the MNZ and PW conclusions.

For readers interested in comparing our paper with the original PW/MNZ results, we also estimate our model using their dataset running up to 1998:2. Our estimation with the original dataset are generally in line with PW’s findings but exhibits higher uncertainty. The most probable break date is at 1973:1, which is the break date specified by PW. The trend variance has a posterior mode at 0.43 but a second, lower mode at 0.9. The existence of
a structural break is positively supported and the fixed break at 1973:1 is slightly favored over an uncertain break. More detailed results are reported in Table 3, the lower panel of Table 4 and Appendix B.

We turn to the results using full sample from 1947:1 to 2013:3 in the following sections.

4.1 Posterior for the break date

Figure 1 presents the posterior distribution for all possible break dates in the mean growth rate from 1947:1 to 2013:2. The most likely break date we find is 2006:1, which has an posterior probability of 3.03% among all considered. With the complete sample, a recent break date is much more probable than a break date in the early 1970s. However one can immediately see in Figure 1 that if the sample were restricted to the original MNZ/PW period, a break date around 1973:1 would be strongly preferred.

The rate of productivity slowdown ($d$) we estimate amounts to -0.4104 in terms of quarterly rate, significantly away from zero and larger than what is reported by PW (-0.288). Since PW’s results are based on the data set up to 1998:2, there remains the possibility that the growth rate change we find around 2006:1 is simply a larger one than what they find, rather than the
only one during the post-war period. We will discuss such a possibility in Section 4.4.

4.2 Posterior for the trend shock volatility

[Figure 2 goes about here]

As we are interested in understanding how an uncertain break date affects our inferences on the trend and cycle of U.S. GDP, we place our focus on the standard deviation $\sigma_\eta$ of the trend shock $\eta_t$. If the estimated $\sigma_\eta$ is significantly far from zero, we find evidence supporting the stochastic trend in GDP. If $\sigma_\eta$ is estimated to be small and close to zero, the real GDP may be better described as a broken linear trend process as argued by PW.

Figure 2 reports the Bayesian posterior density for $\sigma_\eta$. The posterior for $\sigma_\eta$ exhibits high uncertainty and slight bimodality. The first posterior mode is at about 0.45, which is a little further away from zero when compared to PW’s estimate. The second mode is at about 0.88, close to the MNZ estimate. There are about 70% of posterior samples for $\sigma_\eta$ centering around the first mode and 30% around the second one. The uncertainty given the finite sample information reveals the limited power of the current data to clearly identify whether or not the stochastic component is important for the U.S. real output.

The posterior density under the three alternative specifications are also reported in Figure 2 for comparison. They are PW (with a break at 1973:1),
PW06 (with a break at 2006:1) and MNZ (no break). Inferences of $\sigma_\eta$ and the associated uncertainty are somewhat sensitive to the break date chosen. Given a fixed break date at 1973:1, we find results similar to the original PW settings. $\sigma_\eta$ has a sharp posterior mode at 0.4, being very unlikely to be as large as the MNZ estimate 1.2368. But for a fixed break date at 2006:1, the most probable break date we find, the posterior for $\sigma_\eta$ is almost the same as when we allow for an uncertain break. The posterior for the no break, MNZ, model is relatively more diffuse, but with most of its mass to the right of the posterior mass for the break date models.

### 4.3 The trend-cycle decomposition

Figure 3 shows the posterior median of the estimated trend component of GDP, with picture limited to recent years for readability. Our trend estimate suggests the stochastic trend has bent down significantly in the last decade, meaning that U.S. real output has experienced a permanent loss during the period. In addition to the median estimated mean growth rate slowdown of -0.4104% per quarter after the structural break, the permanent loss also reflects accumulated ongoing real shocks, which, for example, cause a total growth rate loss of 2% in the real GDP during the last recession\(^7\).

\[\text{[Figure 3 goes about here]}\]

\(^7\)Estimates are based on the median posterior estimator of the trend shock $\eta_t$. 

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We report our decomposed cycles in Figure 4. Estimated cycles from PW with a fixed break at 2006:1 and MNZ models are also presented for comparisons. Although the Great Recession has been widely considered the most severe in the post-war period, our results suggest the low point of the cycle was -3.013%, which is not so deep as several earlier troughs. In fact, none of the models reported in this paper predicts an extremely large negative cycle despite of different opinions on the overall trend levels and the recovery paths.

It has also been a concern that the economic recovery from the past recession has been extremely slow since the June 2009 NBER end date. Our results suggest that the current GDP level is close to the current trend.

4.4 Robustness checks

We also consider a two-break version of our model for a robustness check. When we allow two structural breaks in the mean growth rate, we find one structural break around 1968:3, close to the original PW date, and the second at 2006:1. However, the log marginal likelihood of the two break model is -359.048, lower than the one-break version. We interpret this as very mild evidence in favor of a single break. Details of this model are reported in Appendix C.
PW also consider a multiple break model which is a variant of Clark (1987). Rather than a single break, the mean growth rate itself follows a random walk. In order to model a rare, large change in mean, PW model the evolution of the mean growth rate as a mixture of two normals with the restriction that most draws (more than 90 percent) come from a normal with an extremely small variance. The PW maximum likelihood estimates essentially indicate only a single, noticeable break. We apply our Bayesian approach to this extended PW model for our longer sample. We do add one change: we allow for correlation between trend and cycle shocks for the reasons outlined in MNZ and in Oh and Zivot (2006), and because a Bayesian comparison mildly prefers the correlated shock specification. The median estimate of the trend variance is essentially the same as in our benchmark model, although with a long right tail. Median estimates of the trend slope function show a major decrease from 1998 to 2008, after a period of much milder slowdown between 1965 and 1990. Details appear in Appendix D.

5 Conclusion

We conduct Bayesian model averaging to endogenize break date uncertainty for the trend-cycle decomposition of U.S. real GDP. We find positive to strong evidence for a structural break in the mean growth rate of the U.S. real output, most likely taking place around 2006:1, based on our Bayesian estimation. While our estimation results agrees with PW’s finding that the trend
variance is likely to be small, the data does not definitively settle the question, as the posterior shows non-trivial probability of a higher trend variance.

We find that the decline in mean trend GDP growth is larger than previous estimates. The median estimate of the drop in trend growth is large enough to be alarming. The combination of accounting for a break in trend growth together with a series of negative shocks to trend during the Great Recession suggests that the present lackluster state of the economy largely reflects trend rather than a significantly negative point in the cycle.

In summary, although we find a different break date, the evidence favors the position of PW that once a single structural break is accounted for the variance of the trend component of GDP is relatively small. However, the evidence is much less than decisive.

References


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Figure 1: Posterior distribution of break dates.

Note: All (including both dark and light) shaded areas represent the 90% HPD regions, while the dark ones represent the 70% HPD regions.
Posterior density for $\sigma_{\eta}$

Figure 2: Posterior density for $\sigma_{\eta}$. 
Figure 3: Posterior estimate of the trend for the benchmark model.  
Note: Shaded areas represent the NBER recession periods.
Figure 4: Posterior estimate of cycles for the benchmark model.
Note: Shaded areas represent the NBER recession periods.
<table>
<thead>
<tr>
<th>$2\log(B_{ij})$</th>
<th>Evidence for $M_i$, against $M_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 2</td>
<td>Not worth more than a bare mention</td>
</tr>
<tr>
<td>2 to 6</td>
<td>Positive</td>
</tr>
<tr>
<td>6 to 10</td>
<td>Strong</td>
</tr>
<tr>
<td>&gt; 10</td>
<td>Very strong</td>
</tr>
</tbody>
</table>

Table 1: Criteria for model comparisons based on twice log Bayes factor
### Benchmark model: with an uncertain break date

Log marginal likelihood: -358.4326

<table>
<thead>
<tr>
<th>Prior</th>
<th>Posterior</th>
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<tbody>
<tr>
<td>median 90% quantile</td>
<td>mean median 90% HPD*</td>
</tr>
<tr>
<td>( \mu )</td>
<td>1</td>
</tr>
<tr>
<td>( \phi_1 )</td>
<td>1</td>
</tr>
<tr>
<td>( \phi_2 )</td>
<td>-0.5</td>
</tr>
<tr>
<td>( \sigma_\eta )</td>
<td>0.24</td>
</tr>
<tr>
<td>( \sigma_\epsilon )</td>
<td>0.61</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0</td>
</tr>
<tr>
<td>( d )</td>
<td>0</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Posterior***</th>
</tr>
</thead>
<tbody>
<tr>
<td>PW06 model: fixed break date</td>
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<tr>
<td>Log marginal likelihood: -356.2768</td>
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<tr>
<td>mean median 90% HPD</td>
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<tr>
<td>( \mu )</td>
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<tr>
<td>( \phi_1 )</td>
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<td>( \phi_2 )</td>
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<td>( \sigma_\eta )</td>
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<tr>
<td>( \rho )</td>
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<td>( d )</td>
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<td>( Tb )</td>
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| MNZ model: no break date |
| Log marginal likelihood: -360.8435 |
| mean median 90% HPD |
| \( \mu \) | 0.8574 | 0.8589 | (0.79, 0.93) | 0.7946 | 0.7975 | (0.69, 0.89) |
| \( \phi_1 \) | 1.4834 | 1.5172 | (1.31, 1.71) | 1.4530 | 1.5016 | (1.22, 1.74) |
| \( \phi_2 \) | -0.5801 | -0.5924 | (-0.76, -0.43) | -0.5467 | -0.5650 | (-0.75, -0.38) |
| \( \sigma_\eta \) | 0.6324 | 0.5601 | (0.23, 1.13) | 0.8976 | 0.9113 | (0.33, 1.39) |
| \( \sigma_\epsilon \) | 0.6200 | 0.5799 | (0.32, 0.92) | 0.8195 | 0.8231 | (0.35, 1.20) |
| \( \rho \) | 0.1816 | 0.2444 | (-0.93, 0.04)∪ | (-0.96, -0.09)∪ | (-0.93, 0.04)∪ | (-0.96, -0.09)∪ |
| \( d \) | -0.6339 | -0.6558 | (-0.90, -0.36) | — | — |
| \( Tb \) | 2006:1 (fixed) | — | — | — | — |

* HPD refers to highest posterior density regions.

** We report the 70% and 90% HPD for Tb in Figure 1.

*** Priors for the unrestricted parameters in PW and MNZ models are the same as those in the benchmark.

Table 2: Bayesian Inferences with RGDP from 1947:1 to 2013:3.
## Benchmark model: with an uncertain break date

Log marginal likelihood: -298.4713

<table>
<thead>
<tr>
<th>Prior</th>
<th>Posterior</th>
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<tbody>
<tr>
<td>median</td>
<td>90% quantile</td>
</tr>
<tr>
<td>$\mu$</td>
<td>1</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>1</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>-0.5</td>
</tr>
<tr>
<td>$\sigma_\eta$</td>
<td>0.24</td>
</tr>
<tr>
<td>$\sigma_\epsilon$</td>
<td>0.61</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0</td>
</tr>
<tr>
<td>$d$</td>
<td>0</td>
</tr>
</tbody>
</table>

### PW model: fixed break date

Log marginal likelihood: -297.5977

<table>
<thead>
<tr>
<th>mean</th>
<th>median</th>
<th>90% HPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.9662</td>
<td>0.9666</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>1.4655</td>
<td>1.4811</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>-0.5680</td>
<td>-0.5759</td>
</tr>
<tr>
<td>$\sigma_\eta$</td>
<td>0.5330</td>
<td>0.4850</td>
</tr>
<tr>
<td>$\sigma_\epsilon$</td>
<td>0.6362</td>
<td>0.6078</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.4825</td>
<td>0.7717</td>
</tr>
<tr>
<td>$d$</td>
<td>-0.3138</td>
<td>-0.3164</td>
</tr>
<tr>
<td>$T_b$</td>
<td>1973:1 (fixed)</td>
<td>—</td>
</tr>
</tbody>
</table>

### MNZ model: no break date

Log marginal likelihood: -299.7812

<table>
<thead>
<tr>
<th>mean</th>
<th>median</th>
<th>90% HPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.9662</td>
<td>0.9666</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>1.4655</td>
<td>1.4811</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>-0.5680</td>
<td>-0.5759</td>
</tr>
<tr>
<td>$\sigma_\eta$</td>
<td>0.5330</td>
<td>0.4850</td>
</tr>
<tr>
<td>$\sigma_\epsilon$</td>
<td>0.6362</td>
<td>0.6078</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.4825</td>
<td>0.7717</td>
</tr>
<tr>
<td>$d$</td>
<td>-0.3138</td>
<td>-0.3164</td>
</tr>
<tr>
<td>$T_b$</td>
<td>1973:1 (fixed)</td>
<td>—</td>
</tr>
</tbody>
</table>

* HPD refers to highest posterior density regions.
** We report the 70% and 90% HPD for Tb in Appendix B.
*** Priors for the unrestricted parameters in PW and MNZ models are the same as those in the benchmark.

Table 3: Bayesian Inferences with data from 1947:1 to 1998:2.
### Table 4: Log marginal likelihoods and model comparisons.

<table>
<thead>
<tr>
<th>Model</th>
<th>Log marginal likelihood</th>
<th>(2\log(B_{ij}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(Row over column)</td>
</tr>
<tr>
<td><strong>Sample period: 1947:1-2013:3</strong></td>
<td></td>
<td>PW</td>
</tr>
<tr>
<td>uncertain break date (Ours)</td>
<td>-358.4326</td>
<td>-4.3116</td>
</tr>
<tr>
<td>(T_b=2006:1) (PW06)</td>
<td>-356.2768</td>
<td>9.1334</td>
</tr>
<tr>
<td>no break (MNZ)</td>
<td>-360.8435</td>
<td></td>
</tr>
<tr>
<td><strong>Sample period: 1947:1-1998:2</strong></td>
<td></td>
<td>PW</td>
</tr>
<tr>
<td>uncertain break date (Ours)</td>
<td>-298.4713</td>
<td>-1.7472</td>
</tr>
<tr>
<td>(T_b=1973:1) (PW)</td>
<td>-297.5977</td>
<td>4.3670</td>
</tr>
<tr>
<td>no break (MNZ)</td>
<td>-299.7812</td>
<td></td>
</tr>
</tbody>
</table>
Appendices (Not for Publication.)

A  The Gibbs sampling used in Section 3.1

Define $\theta = \{\mu, \phi_1, \phi_2, \sigma_\eta, \sigma_\epsilon, \rho, d\}$. Let $(\cdot)^{(k)}$ denote the the $k^{th}$ posterior draw of the latent variable $x_t$ or the parameters. $Y$ denotes all the observed quarterly log real GDP $\{y_1, y_2, \ldots y_T\}$. The $k^{th}$ step in our Gibbs sampler used in Section 3.1 involves the following blocks:

- Draw $\{x_t^{(k)} : t = 1, \ldots T\} \sim f(x_1, \ldots x_T|Y, \theta^{(k-1)}, T h^{(k-1)})$ obtained from the simulation smoother developed by Durbin and Koopman (2002). We then obtain $\tau_t^{(k)}$ and $c_t^{(k)}$ as the first two elements in $x_t^{(k)}$, as well as the residual terms $[\hat{\eta}_t, \hat{\epsilon}_t]^\prime$.

- Draw $[\phi_1^{(k)}, \phi_2^{(k)}] \sim f(\phi_1, \phi_2|Y, x_t^{(k)}, \sigma_\epsilon^{(k-1)})$ given that the second row in (7) has the following regression form:

$$c_t = \begin{bmatrix} c_{t-1} \\ c_{t-2} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \epsilon_t$$

(10)

Stack (10) by time, we have

$$Y_c = C\Phi + \epsilon$$

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where $Y_c = \left\{ c_3^{(k)}, \ldots , c_T^{(k)} \right\}'$, $\epsilon = \{ \epsilon_3 , \ldots , \epsilon_T \}' \sim N(0, 1/h_\epsilon)$ with $h_\epsilon = (\sigma_\epsilon^{(k-1)})^{-2}$, $\Phi = \{ \phi_1 + \phi_2, \phi_2 \}'$ and

$$\begin{bmatrix}
  c_2 & c_1 - c_2 \\
  \vdots \\
  c_{t-1} & c_{t-2} - c_{t-1} \\
  \vdots \\
  c_{T-1} & c_{T-2} - c_{T-1}
\end{bmatrix}$$

Given multivariate normal prior $N(\Phi_0, V_{\Phi0})$ for $\Phi$, the posterior distribution follows a multivariate normal distribution $N(\tilde{\Phi}, \tilde{V}_\Phi)$, where

$$\tilde{V}_\Phi = (V_{\Phi0}^{-1} + h_\epsilon C' C)^{-1} \quad (11)$$

$$\tilde{\Phi} = \tilde{V}_\Phi (V_{\Phi0}^{-1} \Phi_0 + h_\epsilon C' Y_c) \quad (12)$$

The posterior samples for $[\phi_1^{(k)}, \phi_2^{(k)}]$ must guarantee the stationarity of the process. Therefore, we discard nonstationary draws and regenerate new ones until they meet the stationary requirement.

- Draw $[\mu^{(k)}, d^{(k)}] \sim f(\mu, d | Y, x_1^{(k)}, \sigma_\eta^{(k-1)})$ given the regression in the first row of (7):
\[
\tau_t - \tau_{t-1} = \begin{bmatrix} 1 & 1(t > T_b) \end{bmatrix} \begin{bmatrix} \mu \\ d \end{bmatrix} + \eta_t \quad (13)
\]

Stack (13) by time, we have

\[
Y_\tau = D \begin{bmatrix} \mu \\ d \end{bmatrix} + \eta
\]

where \( Y_\tau = \{ \tau_2^{(k)} - \tau_1^{(k)}, \ldots, \tau_T^{(k)} - \tau_{T-1}^{(k)} \} \)' \(, \eta = \{ \eta_2, \ldots, \eta_T \} \)' \( \sim \) \( N(0, 1/h_\eta) \)

with \( h_\eta = (\sigma_\eta^{(k-1)})^{-2} \) and

\[
D = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 1(t > T_b) \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix}
\]

Given uniform priors \( \mu \sim [0, 2] \) and \( d \sim [-1, 1] \), the posterior distribution follows a truncated multivariate normal distribution. It’s equivalent to sampling the posterior from \( N(\tilde{M}, \tilde{V}_M) \) and discard the samples.
out of the valid region, where

\[ \tilde{V}_M = (h \eta D'D)^{-1} \]  \hspace{1cm} (14)

\[ \tilde{M} = h \eta \tilde{V}_M D'Y_\tau \]  \hspace{1cm} (15)

• Draw \[ \begin{bmatrix} \sigma_{1}^{(k)} \\ \sigma_{2}^{(k)} \end{bmatrix} \sim f(\sigma_{1}, \sigma_{2}|Y, x_t^{(k)}, \mu^{(k)}, q^{(k)}, \phi^{(k)}_1, \phi^{(k)}_2, b^{(k-1)}) \]. Define \[ \eta_t^* = \eta_t \sim N(0, \sigma_1^2) \] and \[ \epsilon_t^* = -b\eta_t + \epsilon_t \sim N(0, \sigma_2^2) \], and we have the following:

\[ B^{-1} \begin{bmatrix} \hat{\eta}_t \\ \hat{\epsilon}_t \end{bmatrix} = \begin{bmatrix} \hat{\eta}_t \\ -b\hat{\eta}_t + \hat{\epsilon}_t \end{bmatrix} = \begin{bmatrix} \hat{\eta}_t^* \\ \hat{\epsilon}_t^* \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right) \]  \hspace{1cm} (16)

where

\[ B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \]

According to (16),
\[
\begin{bmatrix}
\eta^*_t \\
\epsilon^*_t
\end{bmatrix}
\sim N\left(
\begin{bmatrix} 0 \\ 0 \end{bmatrix},
\begin{bmatrix} \sigma^2_1 & 0 \\
0 & \sigma^2_2
\end{bmatrix}\right)
\]  

(17)

We assume independent priors for \(h_i^{-1} = \sigma^2_i \sim IG(s_{i0}^{-2}, \nu_{i0})\) with \(i = 1, 2\). It’s equivalent to assume a Gamma prior \(G(s_{i0}^{-2}, \nu_{i0})\) for \(h_i\). The posterior of \(h_i\), in this case, is \(G(\tilde{s}_i^{-2}, \tilde{\nu}_i)\), where for \(i = 1, 2\)[437]

\[
\tilde{\nu}_i = T + \nu_{i0}
\]

(18)

\[
\tilde{s}_1^2 = \eta^*\eta^* + \nu_{10}s_{10}^2
\]

(19)

\[
\tilde{s}_2^2 = \epsilon^*\epsilon^* + \nu_{20}s_{20}^2
\]

(20)

\[
\eta^* = [\eta^{*(k)}_1, ..., \eta^{*(k)}_T]'
\]

(21)

\[
\epsilon^* = [\epsilon^{*(k)}_1, ..., \epsilon^{*(k)}_T]'
\]

(22)

• Draw \(b^{(k)} \sim f(b|Y, x^{(k)}_t, \mu^{(k)}, d^{(k)}, \phi^{(k)}_1, \phi^{(k)}_2, \sigma^2_2)\). Given the second row in (16), we have a standard regression to sample \(b\):

\[
\hat{\epsilon}_t = \hat{\eta}b + \epsilon^*_t
\]

(23)

where \(\epsilon^*_t \sim N(0, \sigma^2_2)\).
Stack (23) by time, we have

\[ \hat{\epsilon} = Eb + \epsilon^* \]

where \( \hat{\epsilon} = \{ \hat{\epsilon}_1^{(k)}, ..., \hat{\epsilon}_T^{(k)} \}' \), \( E = \{ \hat{\eta}_1^{(k)}, ..., \hat{\eta}_T^{(k)} \}' \) and \( \epsilon^* = \{ \epsilon_1^*, ..., \epsilon_T^* \}' \sim \text{i.i.d.} \)

\( N(0, 1/h_{s2}) \) with \( h_{s2} = (\sigma_2^{(k)})^{-2} \). \( \hat{\eta}_i^{(k)} \) and \( \hat{\epsilon}_i^{(k)} \) are residuals in the first two rows in (7).

Given normal prior \( N(b_0, V_{b0}) \) for \( b \), the posterior distribution follows a normal distribution \( N(\hat{b}, \hat{V}_{b}) \), where

\[ \hat{V}_{b} = (V_{b0}^{-1} + h_{s2}E'E)^{-1} \]  
\[ \hat{b} = \hat{V}_{b}(V_{b0}^{-1}b_0 + h_{s2}E'\hat{\epsilon}) \]  

(24)  
(25)

- Draw \( Tb^{(k)} \) \(~f(Tb|Y, \theta^{(k)})\). According to Wang and Zivot (2000), given the flat proper prior assumed for \( Tb \),

\[ f(Tb|Y, \theta) = \frac{f(Y|Tb, \theta)f(Tb|\theta)}{f(Y|\theta)} \]

\( \propto f(Y|Tb, \theta)f(Tb) \)

\( \propto f(Y|Tb, \theta) \)  

(26)
According to (26), we can draw $Tb$ from a multinomial distribution

where

$$f(Tb|Y, \theta) = \frac{f(Y|Tb, \theta)}{\sum_{t=1}^{T-1} f(Y|Tb=t, \theta)}$$

Note that the fixed break date and the no break date model can be also estimated by the above Gibbs sampler with corresponding blocks skipped for the restricted parameters.

**B Results with dataset up to 1998:2**

![Posterior probability for each break date](image)

**Figure 5:** Posterior distribution of break dates.

Note: All (including both dark and light) shaded areas represent the 90% HPD regions, while the dark ones represent the 70% HPD regions.
Figure 6: Posterior density of $\sigma_\eta$. 
C Results with two uncertain break dates

By assuming that there are two unknown structural breaks in the mean growth rate, we assume that

\[ \mu_t = \mu + 1(t > T_{b1})d_1 + 1(t > T_{b2})d_2 \]  \tag{27} \]

For the priors of two break dates, we assume a discrete uniform prior for them over all ordered subsequences of length 2. This prior assumes that all combinations of two break dates are equally likely. For the first (second) break date, its posterior on each possible date in \([1, T_{b2} - 1] \) (\([T_{b1} + 1, T - 1]\)) is proportional to the likelihood conditional on other parameters and the other break date. Other priors and estimation approaches are the same as described in Appendix A.

The posterior mode of the two break dates are 1969:3 and 2006:1 respectively. Posterior density of \( \sigma_\eta \) centers more on the small end and there is no significant sign of bimodality. Other results, such as the trend-cycle decomposition, in general don’t change much from the one-break case. The log marginal likelihood of this case is -359.048, lower than the one-break case.
Figure 7: Posterior distribution of break dates. 
Note: Shaded areas represent the 90% HPD regions.

Figure 8: Posterior density of $\sigma_\eta$. 

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Figure 9: Posterior estimate of cycles for the model with two uncertain breaks.

Note: Shaded areas represent the NBER recession periods.
D Bayesian estimation of the extended PW model

To allow the mean growth rate to change in an uncertain date, Perron and Wada (2009) proposed the following model in addition to their fixed break date model:

\[ y_t = \tau_t + c_t + \omega_t \] (28)

\[ \tau_t = \beta_t + \tau_{t-1} + \eta_t \] (29)

\[ \beta_t = \beta_{t-1} + \nu_t \] (30)

\[ c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \epsilon_t \] (31)

where \( y_t, \tau_t, c_t \) and \( \omega_t \) are the quarterly log RGDP, trend, cycle and a term capturing outliers. The trend slope \( \beta_t \) (equivalent to \( \mu_t \) in our benchmark model) follows a random walk process. And the cycle component follows an AR(2) process.

Further, they assume that two shocks to the systems, \( \nu_t \) and \( \epsilon_t \) are mixtures of normal distributions in order to allow for a large, but rare, growth rate change and potential asymmetric cycles. Specifically, they assume
\[ \nu_t = \lambda_t \gamma_{1t} + (1 - \lambda_t) \gamma_{2t} \quad \text{(32)} \]
\[ \epsilon_t = \delta_t \xi_{1t} + (1 - \delta_t) \xi_{2t} \quad \text{(33)} \]

where \( \gamma_{it} \sim i.i.d. N(0, \sigma_{\gamma}^2) \) and \( \xi_{it} \sim i.i.d. N(0, \sigma_{\xi}^2) \). \( \lambda_t \) and \( \delta_t \) are independent Bernoulli processes that take value 1 with probability \( \alpha_1 \) and \( \alpha_2 \). They assume zero correlation between the trend and cycle shocks. For identification they assume that \( \alpha_1 > 0.9, \sigma_{\gamma_1}^2 < \sigma_{\gamma_2}^2, \sigma_{\xi_1}^2 < \sigma_{\xi_2}^2 \) and \( \sigma_{\gamma_1}^2 < 0.0001 \).

We make two changes to their model when conducting the Bayesian estimation. First, we allow the trend and cycle shocks to be correlated for the reasons outlined in Morley et al. (2003) and Oh and Zivot (2006). As we again adopt the decomposition of (8), we will let \( \sigma_\epsilon \) follow a process similar to \( \sigma \) and \( b \) be constant. The resulting correlation switches between two states controlled by \( \delta_t \), being always in the same state as \( \sigma_\epsilon \) does. Posterior odds of the model with the original zero correlation setting versus the one with correlation can be computed by the Savage-Dickey ratio (See Koop (2003) for reference) if assuming equal prior probability for models. Second, we drop the assumption of \( \sigma_{\gamma_1}^2 < 0.0001 \) as differences are negligible for the priors we try.

As this is a state-space model with a special case of Markov-switching process, we follow the Bayesian estimation approach described in Kim and Nelson (1999). Priors for parameters are chosen exactly the same as with
the benchmark model for those shared by both models, or following common choices in the literature otherwise. Prior and Posterior medians and moments are reported in Table 5. And posterior density of the trend volatility $\sigma_\eta$ and the estimated trend slope function $\beta_t$ are reported in Figures 10 and 11. Generally speaking, results are in line with what we find with our benchmark model. The posterior odds of the model without versus the one with trend-cycle correlation is estimated to be 0.8781, mildly favoring the existence of trend-cycle correlation.

<table>
<thead>
<tr>
<th></th>
<th>Priors median (90% quantiles)</th>
<th>Posterior median (90% HPD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_\eta$</td>
<td>0.24 (0.05, 24.45)</td>
<td>0.47 (0.08, 1.20)</td>
</tr>
<tr>
<td>$\sigma_{\xi_1}$</td>
<td>0.61 (0.08, 83.62)</td>
<td>0.45 (0.13, 1.40)</td>
</tr>
<tr>
<td>$\sigma_{\xi_2}$</td>
<td>0.74 (0.09, 93.77)</td>
<td>1.28 (0.95, 2.04)</td>
</tr>
<tr>
<td>$\sigma_{\gamma_1}$</td>
<td>0.24 (0.05, 24.45)</td>
<td>0.02 (0.01, 0.07)</td>
</tr>
<tr>
<td>$\sigma_{\gamma_2}$</td>
<td>0.24 (0.05, 24.45)</td>
<td>0.17 (0.05, 0.75)</td>
</tr>
<tr>
<td>$\sigma_\omega$</td>
<td>0.24 (0.05, 24.45)</td>
<td>0.19 (0.05, 0.36)</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>1 (-1.32, 2.32)</td>
<td>1.38 (1.11, 1.55)</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>-0.50 (-2.14, 1.14)</td>
<td>-0.47 (-0.63, -0.30)</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>1 (0.90 (restricted), 1)</td>
<td>0.99 (0.96, 1.00)</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>1 (0.52, 1)</td>
<td>0.75 (0.54, 1.00)</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>0 (-0.99, 0.99)</td>
<td>-0.70 (-0.99, 0.95)</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>0 (-0.99, 0.99)</td>
<td>-0.17 (-0.99, 0.21)</td>
</tr>
</tbody>
</table>

Table 5: Bayesian priors and posteriors for the extended model in Perron and Wada (2009) Section 5.
Figure 10: Posterior density of trend shock volatility $\sigma_\eta$. 
Figure 11: Posterior estimate of the slope of the trend function.