Risk Aversion and Stock Price Volatility

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Abstract

Researchers on variance bounds tests of stock price volatility recognized early that risk aversion can increase the price volatility implied by the present-value model. This finding suggests that specifying risk neutrality may induce a bias toward rejecting the present-value model insofar as real-world investors are risk averse. However, establishing that risk aversion may increase stock price volatility does not, by itself, have implications for the presence or absence of excess volatility. This is so because risk aversion also affects the upper-bound volatility measure computed from “perfect foresight” (or “ex-post rational”) stock prices. Consequently, while high risk aversion implies high volatility in some settings, it does not necessarily imply excess volatility. This paper compares price volatility computed from real-world data to model-predicted volatility measures in a setting that allows for risk aversion. Using variance bounds tests based on the price-dividend ratio, we find evidence of excess volatility in U.S. stock prices for relative risk aversion coefficients below 5. For higher degrees of risk aversion, the evidence for excess volatility is less clear. We also ask whether variance bounds involving returns can be established in settings involving risk aversion and autocorrelated dividend growth. We show that the answer is no. Except in special cases, the present-value model does not impose bounds on return volatility in our setting.

Keywords: Asset Pricing, Excess Volatility, Variance Bounds, Risk Aversion, Present-Value Model.

JEL Classification: E44, G12.

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The variance-bounds tests of stock price volatility reported in the mid-1970s and later raised doubts about the conclusion (then generally accepted) that stock prices can be accurately represented as the present value of expected future dividends with a constant discount factor. Price and payoff volatility in the data both appeared to exceed the values implied by the discounted-dividend model with a constant discount factor. How were these findings to be reconciled with the conclusions of the earlier efficient markets literature supporting the discounted-dividend model? A number of econometric problems with the variance-bounds tests were unearthed, but it turned out that correcting these problems did not eliminate the appearance of excess volatility.¹

The debate ended—somewhat abruptly—in the 1990s when economists using analytical methods unrelated to those employed in the variance-bounds tests began reaching conclusions that reversed those of the earlier efficient-markets literature, but were consistent with the findings of the variance-bounds tests. Campbell and Shiller (1988), for example, employed a log-linear decomposition of the return identity that turned out to be extremely fruitful. Using this decomposition, they resolved stock prices into terms consisting of expected future dividends and expected future returns. They rejected the notion that expected returns are governed by a white noise process, as implied by the constant-discount-factor version of the present-value model. Instead, they found that expected returns include an autoregressive component that exhibits high persistence. Even if the innovations to the expected return process are small, the presence of the autoregressive component can greatly increase implied stock price volatility. Thus there was no longer a discrepancy between the conclusions of the variance-bounds tests and those reached using other methods.²

This paper, like that of Campbell and Shiller, makes use of log-linear methods to examine stock price volatility. However, rather than log-linearizing the return identity as Campbell and Shiller did, we compute a log-linear approximation of the representative agent’s first-order condition. This approximation incorporates a model specification for the stochastic discount factor and a process for consumption/dividends.

The fact that two distinct analytical procedures reach similar conclusions leaves open the question of which method is to be preferred for research on asset pricing. Some take the view that the Campbell-Shiller method, which is not tied to a particular asset pricing model, is less prone to misspecification. Perhaps so, but the fact that analysts using variance-bounds tests came to the same (now generally-accepted) conclusion much earlier than others suggests that the variance-bounds tests focus attention on essential considerations that might otherwise be overlooked.

¹For summaries of this large literature, see West (1988a), Gilles and LeRoy (1991) and Shiller (2003).
²As discussed in LeRoy (2010), the finding that expected returns have an autoregressive component is sometimes interpreted to be an explanation for apparent excess volatility of stock prices and returns. That interpretation is incorrect. In the absence of a viable theory for the autoregressive component (which appears unrelated to potential explanatory variables such as interest rates), the finding of an autoregressive component in expected returns should be interpreted as a restatement of the excess volatility conclusion, not an explanation for it.
Let us see what these considerations might be. A fact that is often glossed over in discussions of stock market efficiency is that the proposition being tested is a compound null hypothesis. Stock prices are taken to equal the present value of future dividends with the univariate (i.e., marginal) distribution of dividends taken as given, but the null hypothesis is silent about how much information investors condition on when forming their expectations of future dividends. For example, LeRoy and Porter (1981) thought of dividends as generated jointly with other unspecified variables as a multivariate ARMA process, which leaves open the possibility that other variables—current earnings, for example—are predictors of future dividend innovations. Assuming that investors are risk neutral, an increase in the amount of information that investors are assumed to have about future dividend innovations will raise the unconditional variance of stock prices, but lower the variance of payoffs or price-changes.\(^3\)

The upper bound on unconditional price variance, corresponding to a lower bound of zero on payoff variance, is reached when investors are assumed to have perfect foresight about the entire future path of dividends. In contrast, the upper bound on payoff variance is reached in the opposite case when investors have no auxiliary information about future dividend innovations, i.e., when investors only know current and past dividends. Since the unconditional variance of payoffs is nearly the same as the conditional variance of prices, the variance bounds test is effectively a joint test on the unconditional and conditional variances implied by the present-value model.

When economists talk about stock price volatility it is not clear whether unconditional or conditional variance best corresponds to what they have in mind.\(^4\) The frequent use of terms like “choppiness” or “smoothness” in describing stock prices suggests that the conditional variance is the appropriate concept, since these terms are taken to refer to short-term price volatility. The fact just noted that the unconditional and conditional variances are affected in opposite directions by variables measuring the extent of investors’ auxiliary information implies that this lack of clarity is an important problem. We do not take the position here that either of these variance measures is superior to the other; instead we will determine the implications of the present-value model for both.

The point is that a comparison of actual and perfect foresight stock prices, as is inherent in the variance-bounds tests, forces the analyst to focus on the degree to which investors possess information over and above that contained in current and past dividends. The variance bounds are generated by imposing extreme hypothetical specifications for investors’ information. Given the explicit focus on information assumptions, variance-bounds tests remain a useful analytical tool for assessing the success or failure of the present-value model. Of course, we do not suggest restricting attention to these methods.

A major shortcoming of the early variance-bounds tests was the maintained assumption that returns follow a white-noise process. While in stationary settings this assumption can be justified when investors are risk neutral, the time-series behavior of returns will be more com-

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3. This basic result is established in LeRoy and Porter (1981) and West (1988b).

4. LeRoy (1984) and Kleidon (1986) use numerical examples to illustrate the idea that different conclusions may be drawn from considering conditional variances rather than unconditional variances.
plex than a white-noise process if investors are risk averse. The assumption of risk-neutrality in the early variance-bounds tests appeared acceptable at the time because the empirical literature on stock market efficiency had been based on exactly the same assumption, although this was seldom made explicit. However, the development of quantitative consumption-based asset pricing models, as manifested in such papers as Mehra and Prescott (1985), led analysts to a heightened appreciation of the potential importance of risk-aversion.

Risk aversion was recognized early as a possible cause for findings of apparent excess volatility of stock prices (see Grossman and Shiller 1981, LeRoy and LaCivita 1981): at least in some settings risk aversion increases the implied volatility of asset prices relative to the risk-neutral case. This is so because risk-averse investors wish to smooth consumption over time. They do this by trying to acquire financial assets in good times and liquidate them otherwise. In such settings, market clearing implies that asset prices will be higher in good times and lower in bad times by a factor that depends on risk aversion, relative to the risk-neutral case.

These arguments, however, were incomplete. Establishing that risk aversion may affect actual stock price volatility does not, by itself, have implications for the presence or absence of excess price volatility. This is so because risk aversion also affects the upper-bound volatility measure computed from “perfect foresight” (or “ex-post rational”) stock prices. Consequently, while high risk aversion implies high price volatility in some settings, it does not necessarily imply excess price volatility. It remained unclear whether the existence of risk aversion could alter the proposition that the variance computed from perfect foresight prices represents an upper bound for the variance computed from actual prices.

This paper compares the price and return volatilities computed from actual data to model-predicted volatility measures in a setting that allows for risk aversion. Using variance bounds tests based on the price-dividend ratio, we find evidence of excess volatility in U.S. stock prices for risk aversion coefficients below about 5. For higher degrees of risk aversion, we find that volatility is not excessive if investors can be assumed to be able to predict dividends into the distant future. To the extent that this assumption is viewed as implausible, it follows that price volatility is excessive in that case as well.

In log-linear settings, as here, return variance is the analogue to the concept of payoff variance or price-change variance examined by LeRoy and Porter (1981) and West (1988b). We show by counterexample that the return variance analogues to the earlier LeRoy-Porter-West results do not apply when investors are risk averse.

1 Variance Bounds under Risk Neutrality

Whether or not agents are risk neutral, the equilibrium stock price $p_t$ obeys $p_t = E(p_t^*|I_t)$, where $I_t$ represents investors’ information about future dividend realizations and $p_t^*$ is the perfect foresight price. As originally set forth in Shiller (1981) and LeRoy and Porter (1981), the fact that $p_t$ equals the conditional expectation of $p_t^*$ implies that the variance of $p_t^*$ is an upper bound for the variance of $p_t$.

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5 See LeRoy (1973) and Lucas (1978).
upper bound for the variance of \( p_t \). As LeRoy and Porter (1981) showed, we can also establish a lower bound on the variance of \( p_t \). Define \( H_t = \{d_t, d_{t-1}, d_{t-2}, \ldots\} \) as the information set consisting only of current and past dividends, and define \( \hat{p}_t = E(p_t^* | H_t) \), where \( \hat{p}_t \) is the appropriate stock price for an econometrician who has no information useful in forecasting \( p_t^* \) other than current and past dividends.\(^7\)

Suppose that investors’ information \( I_t \) contains at least \( H_t \), so that \( H_t \subseteq I_t \), but investors may also have auxiliary information over and above current and past dividends that is useful in predicting \( p_t^* \). For example, current earnings or forecasts of future earnings are likely to help predict future dividends even given current and past dividends. The simplest characterization of a distinction between \( H_t \) and \( I_t \) (to be employed below) would define \( I_t = H_t \cup d_{t+1} \), so that investors can see dividends without error one period ahead. The fact that \( I_t \) is a refinement of \( H_t \) implies that \( \text{Var}(\hat{p}_t) \) is a lower bound for \( \text{Var}(p_t) \). We have

\[
\text{Var}(\hat{p}_t) \leq \text{Var}(p_t) \leq \text{Var}(p_t^*),
\]

where the upper and lower variance bounds, unlike the stock price \( p_t \), can be calculated from a univariate model for dividends. The bounds therefore can be derived without explicitly specifying the extent of investors’ auxiliary information.

We define the “excess payoff” under information set \( I_t \) as

\[
\nu_{t+1} \equiv p_{t+1} + d_{t+1} - \beta^{-1} p_t,
\]

which represents next-period’s cash value from the stock investment minus the payoff from an equal investment in the risk-free asset. Under risk-neutrality, the real risk-free expected gross return on stock equals \( \beta^{-1} \), where \( \beta \) is the investor’s subjective time discount factor.\(^8\)

From the investor’s first-order condition, we have \( p_t = \beta E \{ (p_{t+1} + d_{t+1}) | I_t \} \) which implies that the excess payoff is simply the one-period-ahead forecast error which is iid over time. Multiplying successive iterations of equation (2) by \( \beta^i \) for \( i = 1, 2, 3, \ldots \) and then summing across the resulting equations yields

\[
\beta \nu_{t+1} + \beta^2 \nu_{t+2} + \beta^3 \nu_{t+3} + \ldots = -p_t + \beta d_{t+1} + \beta^2 d_{t+2} + \beta^3 d_{t+3} + \ldots + \hat{p}_t
\]

Solving equation (3) for \( \hat{p}_t \) and then taking the variance of both sides yields

\[
\text{Var}(\hat{p}_t) = \text{Var}(p_t) + \frac{\beta^2}{1 - \beta^2} \text{Var}(\nu_t),
\]

where we have assumed that dividends are generated by a stationary linear process so that the variances are constant. The perfect-foresight version of the first-order condition is \( p_t^* = \)

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\(^7\)Throughout the paper, we adopt the notation of using stars “*” to denote perfect foresight variables, hats “\(^\hat{\}\)” to denote variables computed using information set \( H_t \), and unmarked variables (such as \( p_t \)) to denote variables computed using information set \( I_t \).

\(^8\)Under risk aversion, the risk-free return is not constant but rather depends on expected consumption growth.
\( \beta \left( p_{t+1}^* + d_{t+1} \right) \), which shows that the excess payoff under perfect foresight is zero for all \( t \) such that \( \text{Var}(\nu_t^*) = 0 \). Since \( \text{Var}(p_t^*) - \text{Var}(p_t) \geq 0 \) from (1) the above expression establishes that \( \text{Var}(\nu_t) \geq 0 = \text{Var}(\nu_t^*) \).

Similarly, we define the excess payoff under information set \( H_t \) as \( \widehat{\nu}_{t+1} \equiv \widehat{p}_{t+1} + d_{t+1} - \beta^{-1} p_t \). Following the same methodology as above, we obtain

\[
\text{Var}(p_t^*) = \text{Var}(\widehat{p}_t) + \frac{\beta^2}{1 - \beta^2} \text{Var}(\widehat{\nu}_t). \tag{5}
\]

Substituting for \( \text{Var}(p_t^*) \) from equation (4) into equation (5) and noting that \( \text{Var}(p_t) - \text{Var}(\widehat{p}_t) \geq 0 \) from (1) establishes that \( \text{Var}(\widehat{\nu}_t) \geq \text{Var}(\nu_t) \). Thus, if investors are risk neutral and dividends are generated by a stationary linear process, then the following bounds on payoff variance exist:

\[
\text{Var}(p_{t+1}^* + d_{t+1} - \beta^{-1} p_t) = 0 \leq \text{Var}(p_{t+1} + d_{t+1} - \beta^{-1} p_t) \leq \text{Var}(\widehat{p}_{t+1} + d_{t+1} - \beta^{-1} \widehat{p}_t), \tag{6}
\]

In the above example, the more information agents have about future dividend innovations, the higher is the variance of prices and lower is the variance of excess payoffs. The maintained lower bound on investors’ information is represented by \( H_t \), implying that the payoff variance associated with \( H_t \) is an upper bound for the payoff variance associated with \( I_t \).

In this paper we will use a log-linear rather than linear specification for dividends. In the log-linear setting, the analogue to payoff variance is return variance. We ask whether a return variance analogue to the payoff bounds (6) can be established in settings that allow for risk aversion. We show that the answer is no.

## 2 Asset Pricing Model

Equity shares are priced using the frictionless pure exchange model of Lucas (1978). A representative investor can purchase shares to transfer wealth from one period to another. Each share pays an exogenous stream of stochastic dividends in perpetuity. The representative investor’s problem is to maximize

\[
E \left\{ \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\alpha} - 1}{1 - \alpha} | I_0 \right\}, \tag{7}
\]

where \( \alpha \) is the coefficient of relative risk aversion, \( E \) is the mathematical expectation operator, and \( I_t \) is the investor’s information set at time \( t \). The first-order condition that governs the investor’s share holdings is

\[
p_t = E \left\{ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\alpha} (p_{t+1} + d_{t+1}) | I_t \right\}. \tag{8}
\]
The first-order condition can be iterated forward to substitute out $p_{t+j}$ for $j = 1, 2, \ldots$. Applying the law of iterated expectations and imposing a transversality condition that excludes bubble solutions yields the following expression for the equilibrium stock price:

$$p_t = E \left\{ \sum_{j=1}^{\infty} M_{t,t+j} d_{t+j} | I_t \right\},$$

where $M_{t,t+j} \equiv \beta^j (c_{t+j}/c_t)^{-\alpha}$ is the stochastic discount factor. The perfect foresight price is given by

$$p_t^* = \sum_{j=1}^{\infty} M_{t,t+j} d_{t+j}.$$  

Equity shares are assumed to exist in unit net supply. Market clearing therefore implies $c_t = d_t$ for all $t$.

### 3 Volatility of the Price-Dividend Ratio

Dividends, and therefore also the stock price variables $p_t$ and $p_t^*$, trend upward, implying that dividend and price variances conditional on some initial date will increase with time. To avoid this time-varying volatility result, a trend correction must be imposed. The solution adopted by Shiller (1981) was to assume that dividends and prices are stationary around a time trend. This specification is empirically unrealistic; mean-reversion to a time trend induces negative autocorrelation in growth rates, which conflicts with what we see in long-run U.S. data for the growth rates of real dividends, real stock prices, and post-World War II real per capita consumption.\(^9\)

LeRoy and Porter (1981) corrected for nonstationarity by reversing the effect of earnings retention on dividends and stock prices, but that procedure appeared to produce series that were not stationary.\(^{10}\) Current practice, particularly when using a homothetic utility function like (7), is to correct for trend by working with intensive variables, such as the price-dividend ratio or the rate of return, as these variables will be stationary in the models of interest (see, for example, LeRoy and Parke, 1992). This is the procedure we follow here.

To ensure that equilibrium values of intensive variables are stationary, we assume that the growth rate of dividends $x_t \equiv \log (d_t/d_{t-1})$ is governed by the following AR(1) process

$$x_{t+1} = \bar{x} + \rho (x_t - \bar{x}) + \varepsilon_{t+1}, \quad \varepsilon_{t+j} \sim N (0, \sigma^2) \text{ iid, } |\rho| < 1.$$  

In the special case of $\rho = 0$, the above specification implies that the level of real dividends follows a geometric random walk with drift, as in LeRoy and Parke (1992). The geometric

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10. West (1988a, p. 641) summarizes the various assumptions made in the literature regarding the stationarity of prices and dividends.
random walk model provides a reasonable representation of dividends in the data. However, we do not want to restrict our results to the case where dividend growth rates are iid, so we allow for serially correlated dividend growth ($\rho \neq 0$).

The specification (11) implies that the unconditional moments of dividend growth are given by

$$E(x_t) = \pi,$$  \hspace{1cm} (12)

$$Var(x_t) = \frac{\sigma_x^2}{1-\rho^2},$$  \hspace{1cm} (13)

$$Cov(x_t, x_{t-1}) = \rho Var(x_t).$$  \hspace{1cm} (14)

The price-dividend ratios implied by the information sets $H_t$ and $I_t$ are denoted by $b_p t = d_t$ and $y_t = d_t$, respectively, while the perfect foresight price-dividend ratio is denoted by $y_t = p_t^*/d_t$. By substituting the equilibrium condition $c_t = d_t$ into the first-order condition (8), the first-order condition under the various information assumptions can be written as

$$\hat{y}_t = E\{\beta \exp[(1-\alpha)x_{t+1}] (\hat{y}_{t+1} + 1) | H_t\}$$  \hspace{1cm} (15)

$$y_t = E\{\beta \exp[(1-\alpha)x_{t+1}] (y_{t+1} + 1) | I_t\},$$  \hspace{1cm} (16)

$$y_t^* = \beta \exp[(1-\alpha)x_{t+1}] (y_{t+1}^* + 1).$$  \hspace{1cm} (17)

The fact that $\hat{y}_t$, $y_t$, and $y_t^*$ are ratios with the same denominator $d_t$, together with the fact that $d_t$ is measurable in $H_t$ and $I_t$, immediately implies $Var(\hat{y}_t) \leq Var(y_t) \leq Var(y_t^*)$. Hence the basic form of the variance bound derived in the earlier literature under risk neutrality, i.e., $Var(\hat{p}_t) \leq Var(p_t) \leq Var(p_t^*)$, carries over to the case of risk aversion when the price-dividend ratio (an intensive variable) is substituted for the stock price (an extensive variable).

### 3.1 Variance under Information Set $H_t$

We next obtain an approximate analytical solution for the variance of $\hat{y}_t$ under information set $H_t$. This involves solving the first-order condition (15) subject to the dividend growth process (11). To do so it is convenient to define the following nonlinear change of variables:

$$\tilde{z}_t \equiv \beta \exp[(1-\alpha)x_t] (\hat{y}_t + 1),$$  \hspace{1cm} (18)

where $\tilde{z}_t$ represents a composite variable that depends on both the growth rate of dividends and the price-dividend ratio. The first-order condition (15) becomes $\hat{y}_t = E(\tilde{z}_{t+1}|H_t)$, implying that $\hat{y}_t$ is simply the rational forecast of the composite variable $\tilde{z}_{t+1}$, conditioned on $H_t$. The composite variable $\tilde{z}_t$ is governed by the following equilibrium condition

$$\tilde{z}_t = \beta \exp[(1-\alpha)x_t] [E(\tilde{z}_{t+1}|H_t) + 1],$$  \hspace{1cm} (19)
which shows that the value of \( z_t \) in period \( t \) depends on the conditional forecast of the next-period value of that same variable.

The following proposition presents an approximate analytical solution for the composite variable \( z_t \).

**Proposition 1.** An approximate analytical solution for the equilibrium value of the composite variable \( z_t \) under information set \( H_t \) is given by

\[
\tilde{z}_t = \exp \left[ a_0 + a_1 (x_t - \bar{x}) \right],
\]

where \( a_1 \) solves

\[
a_1 = \frac{1 - \alpha}{1 - \rho \beta \exp \left[ (1 - \alpha) \bar{x} + \frac{1}{2} (a_1)^2 \sigma^2 \right]}
\]

and \( a_0 \equiv E[\log(\tilde{z}_t)] \) is given by

\[
a_0 = \log \left\{ \frac{\beta \exp \left[ (1 - \alpha) \bar{x} \right]}{1 - \beta \exp \left[ (1 - \alpha) \bar{x} + \frac{1}{2} (a_1)^2 \sigma^2 \right]} \right\},
\]

provided that \( \beta \exp \left[ (1 - \alpha) \bar{x} + \frac{1}{2} (a_1)^2 \sigma^2 \right] < 1. \)

**Proof:** See Appendix A.1.

Two values of \( a_1 \) satisfy the nonlinear equation in Proposition 1. The inequality restriction selects the value of \( a_1 \) with lower magnitude to ensure that the point of approximation \( \exp(a_0) \) is positive.\(^{11}\)

Given the approximate solution for the composite variable \( \tilde{z}_t \), we can recover \( \tilde{y}_t \) as follows:

\[
\tilde{y}_t = E(\tilde{z}_t+1|H_t) \approx \exp \left[ a_0 + a_1 \rho (x_t - \bar{x}) + \frac{1}{2} (a_1)^2 \sigma^2 \right]. \tag{20}
\]

As shown in Appendix A.2, the approximate fundamental solution can be used to derive the following unconditional variance of the log price-dividend ratio:

\[
Var[\log(\tilde{y}_t)] = (a_1 \rho)^2 Var(x_t), \tag{21}
\]

which in turn can be used to derive an expression for \( Var(\tilde{y}_t) \).\(^{12}\)

From equation (20), the direction of the effect of fluctuations in dividend growth on \( \tilde{y}_t \) depends on the sign of \( a_1 \), which in turn depends on the value of the coefficient of relative

\(^{11}\)Lansing (2010) compares the approximate solution from Proposition 1 to the exact theoretical solution derived by Burnside (1998). The approximate solution is extremely accurate for low and moderate levels of risk aversion (\( \alpha \approx 2 \)). But even for high levels of risk aversion (\( \alpha \approx 10 \)), the approximation error for the equilibrium price-dividend ratio remains below 5 percent.

\(^{12}\)Given the unconditional mean \( E[\log(\tilde{y}_t)] = a_0 + (a_1)^2 \sigma^2 / 2 \) and the expression for \( Var[\log(\tilde{y}_t)] \) from above, the unconditional variance of \( \tilde{y}_t \) itself can be computed by making use of the following expressions for the mean and variance of the log-normal distribution: \( E(\tilde{y}_t) = \exp \{ E[\log(\tilde{y}_t)] + \frac{1}{2} Var[\log(\tilde{y}_t)] \} \) and \( Var(\tilde{y}_t) = E(\tilde{y}_t)^2 \{ \exp(Var[\log(\tilde{y}_t)]) - 1 \} \).
risk aversion $\alpha$ and the value of $\rho$. Suppose first that $\rho < 0$, so that agents expect that high current dividend growth will be followed by low growth. Assuming $\alpha < 1$ such that $a_1 > 0$, we have $a_1 \rho < 0$ which causes stocks to trade at a lower-than-average multiple of current dividends today, i.e., a lower value of $\hat{y}_t$. On the other hand when $\alpha > 1$ such that $a_1 < 0$, we have $a_1 \rho > 0$. In this case, the expected lower dividend growth in the following period is more than offset by a decline in the realization of the stochastic discount factor, leading to a higher value of $\hat{y}_t$ today. In the intermediate case of logarithmic utility where $\alpha = 1$, fluctuations in dividend growth do not affect $\hat{y}_t$, which is therefore constant. This result obtains because the income and substitution effects of a shock to dividend growth are exactly offsetting. All of these effects are reversed when $\rho > 0$.

Also from equation (20), it is easy to see intuitively how different levels of $\alpha$ affect the variance of $\hat{y}_t$. When $\alpha < 1$, increases in $\alpha$ moves the variance of $\hat{y}_t$ toward zero. This happens because fluctuations in dividend growth are increasingly offset by fluctuations in their marginal utility; the closer $\alpha$ is to unity, the greater is the offset. When $\alpha > 1$, an increase in $\alpha$ raises the extent to which the magnitude of fluctuations in marginal utility exceed the magnitude of fluctuations in inverse consumption, thereby increasing the variance of $\hat{y}_t$.

### 3.2 Variance under Information Set $I_t = H_t \cup d_{t+1}$

In the preceding subsection we assumed that investors have no auxiliary information that would help predict future dividends. We now relax that assumption by allowing investors to see dividends one period ahead, as in LeRoy and Parke (1992). The expanded information set is defined as $I_t = H_t \cup d_{t+1} = \{d_{t+1}, d_t, d_{t-1}, d_{t-2}, \ldots\}$.

As shown in the appendix, the expanded information set $I_t$ implies the following relationships

\[ p_t = M_{t,t+1} (d_{t+1} + \hat{p}_{t+1}), \quad (22) \]

\[ y_t = \beta \exp \left[ (1 - \alpha) x_{t+1} \right] (1 + \hat{y}_{t+1}) = \hat{z}_{t+1}, \quad (23) \]

where $M_{t,t+1}$ is the realized stochastic discount factor at time $t + 1$, but which is known by investors at time $t$. From equations (15) and (16) and the law of iterated expectations, it follows directly that $\hat{y}_t = E(y_t | H_t)$ which in turn implies $\text{Var} (\hat{y}_t) \leq \text{Var} (y_t)$.

From equations (23) and Proposition 1, the approximate law of motion for $y_t = \hat{z}_{t+1}$ implies the following unconditional variance:

\[ \text{Var} [\log (y_t)] = \left( a_1 \right)^2 \text{Var} (x_t). \quad (24) \]

Comparing the above expression to $\text{Var} [\log (\hat{y}_t)]$ from equation (21) shows that $\text{Var} [\log (\hat{y}_t)] \leq \text{Var} [\log (y_t)]$ since $|\rho| < 1$.

\[ ^{13} \]This setup seems particularly realistic in light of company-provided guidance about future financial performance which is typically disseminated to the public via quarterly conference calls.
3.3 Variance under Perfect Foresight

Under perfect foresight, investors are assumed to be able to see all future dividends, not just dividends one period in advance. The perfect foresight price-dividend ratio \( y_t^* \) is governed by equation (17), which is a nonlinear law of motion. To derive an analytical expression for the perfect foresight variance, we approximate equation (17) using the following log-linear law of motion (Appendix C.1):

\[
\log (y_t^*) - E [\log (y_t^*)] \approx (1 - \alpha) (x_{t+1} - \bar{x}) + \beta \exp [(1 - \alpha) \bar{x}] \{ \log (y_{t+1}^*) - E [\log (y_t^*)] \}, \quad (25)
\]

The approximate law of motion (25) and the dividend growth process (11) can be used to derive the following unconditional variance (Appendix C.2):

\[
Var [\log (y_t^*)] = \frac{(1 - \alpha)^2}{1 - \bar{x}^2 \exp [2(1 - \alpha) \bar{x}]} \left\{ \frac{1 + \rho \beta \exp [(1 - \alpha) \bar{x}]}{1 - \rho \beta \exp [(1 - \alpha) \bar{x}]} \right\} Var (x_t), \quad (26)
\]

which is more complicated than either \( Var [\log (\tilde{y}_t)] \) from equation (21) or \( Var [\log (y_t)] \) from equation (24).

3.4 Model Calibration

Given that the Lucas model implies \( c_t = d_t \) in equilibrium, we calibrate the stochastic process for \( x_t \) in equation (11) using U.S. annual data for the growth of per capita real consumption from 1890 to 2004.\(^{14}\) We choose parameters to match the mean, standard deviation, and autocorrelation of consumption growth in the data, using the moment formulas given by equations (12) through (14). This procedure yields \( \bar{x} = 0.0206, \sigma_x = 0.0354, \) and \( \rho = -0.1. \) For each value of \( \alpha, \) we calibrate the subjective time discount factor \( \beta \) so as to achieve \( E [\log (\tilde{y}_t)] = 3.18 \) in the model, consistent with the sample average value of the log price-dividend ratio for the S&P 500 stock index going back to 1871.\(^{15}\) When \( \alpha \) exceeds a value of about 3, achieving the target value of \( E [\log (\tilde{y}_t)] \) in the model requires a value of \( \beta \) that is greater than unity. Nevertheless, for all values of \( \alpha \) examined, the mean value of the stochastic discount factor \( E [\beta (c_{t+1}/c_t)^{-\alpha}] \) remains below unity.\(^{16}\)

3.5 Quantitative Analysis

This section presents a quantitative analysis of the volatility of the log price-dividend ratio in the calibrated version of the Lucas model described above. The top panel of Figure 1 compares the variance of the log price-dividend ratio for the S&P 500 index (solid green line) with the


\(^{15}\)Cochrane (1992) employs a similar calibration procedure. For a given discount factor, he chooses the risk coefficient \( \alpha \) to match the mean price-dividend ratio in the data.

\(^{16}\)Kocherlakota (1990) shows that a well-defined competitive equilibrium with positive interest rates can still exist in growth economies when \( \beta > 1. \)
computed volatility of \( \log(\hat{y}_t) \) predicted by the model (solid blue line) and that of \( \log(y^*_t) \) predicted by the model (dashed red line). The figure also shows the computed volatility of \( \log(y_t) \) predicted by the model (dashed grey line) for the case where \( I_t = H_t \cup d_{t+1} \).

The standard deviation of \( \log(\hat{y}_t) \) is close to zero for all values of \( \alpha \). This low figure reflects the fact that the calibrated autocorrelation of dividend growth rates \( \rho \) is nearly zero, corresponding to a near-geometric random walk in the level of dividends. The model-predicted volatility for \( \log(\hat{y}_t) \) is much lower than the standard deviation of the log price-dividend ratio in U.S. data, which is around 0.4. This finding suggests the presence of excess volatility of stock prices in the data, but it does not conclusively demonstrate its existence because of the possibility that real-world investors may use information variables to predict future dividend growth innovations, which would increase the volatility of the U.S. ratio.

A finding of excess volatility requires \( \text{Var}[\log(y^*_t)] > \text{Var}[\log(y_t)] \). Figure 1 shows that excess volatility prevails for \( \alpha < 5 \). In contrast, for \( \alpha > 5 \), we have \( \text{Var}[\log(y^*_t)] < \text{Var}[\log(y_t)] \), so we cannot make a finding of excess volatility. The interpretation is that the volatility of \( \log(y^*_t) \) is consistent with the present-value model if real-world investors are risk averse and have access to quite good (but not necessarily perfect) information about future dividend growth. The conclusion that the theoretical variance inequality is not satisfied when risk aversion is low is consistent with the early variance-bounds tests, which found excess volatility under the assumption of risk neutrality. Whether one is willing to conclude that volatility is excessive depends on whether one views risk aversion coefficients around 5 as realistic (most empirical estimates are more like 2), and also on whether one finds the assumption that investors can predict dividends into the distant future to be reasonable.

### 4 Return Volatility

We observed in the introduction that notions of price volatility can be connected either with unconditional variance measures, corresponding to a long-run interpretation of volatility, or with conditional variance measures, corresponding to a short-run conception. We also noted that, based on earlier research assuming risk neutrality, the present-value model has implications for both measures of volatility. Specifically, the variance-bounds tests involve determining whether the joint restrictions implied by the present-value model for both volatility measures are satisfied. So far we have concentrated on bounds of unconditional price volatility as embodied in \( \text{Var}[\log(\hat{y}_t)] \) and \( \text{Var}[\log(y_t)] \).

We now turn to measures of short-run price volatility. There are several ways to gauge short-run volatility: the variance of one-period payoffs, the unconditional variance of the log price change, or the unconditional variance of the rate of return. Since these measures are highly correlated, it does not matter much for the substantive results which measure is used.\(^{17}\) It turns out that, just as the variance of one-period payoffs is the most convenient measure of

\(^{17}\)Over the period 1871 to 2008, the correlation coefficient between log real equity returns and log real price changes in U.S. data is 0.994. LeRoy (1984, p. 186) shows that the conditional price variance is numerically very close to the unconditional variance of price changes in a calibrated asset pricing model.
short-run volatility when dividends are governed by a stationary linear process, the variance
of log returns is the most convenient in the log-linear setting considered here.

The gross rates of return on equity under the various information assumptions can be
written as

\[
\hat{R}_{t+1} = \frac{\hat{p}_{t+1} + d_{t+1}}{\hat{p}_t} = \exp \left( \frac{\hat{y}_{t+1} + 1}{\hat{y}_t} \right) \\
= \beta^{-1} \exp (\alpha x_{t+1}) \left[ \frac{\hat{z}_{t+1}}{E(\hat{z}_{t+1}|H_t)} \right], \quad (27)
\]

\[
R_{t+1} = \frac{p_{t+1} + d_{t+1}}{p_t} = \exp (x_{t+1}) \left( \frac{y_{t+1} + 1}{y_t} \right) \\
= \beta^{-1} \exp (\alpha x_{t+1}) \left[ \frac{z_{t+1}}{E(z_{t+1}|I_t)} \right], \quad (28)
\]

\[
R^*_{t+1} = \frac{p^*_{t+1} + d_{t+1}}{p^*_t} = \exp (x_{t+1}) \left( \frac{y^*_{t+1} + 1}{y^*_t} \right) \\
= \beta^{-1} \exp (\alpha x_{t+1}). \quad (29)
\]

In the expressions for \( \hat{R}_{t+1} \) and \( R^*_{t+1} \), we have eliminated \( \hat{y}_t \) and \( y_t \) using the equilibrium
conditions \( \hat{y}_t = E(\hat{z}_{t+1}|H_t) \) and \( y_t = E(z_{t+1}|I_t) \). We have eliminated \( \hat{y}_{t+1} \) using the de-
finitional relationship \( \hat{y}_{t+1} + 1 = \beta^{-1} \exp [-(1 - \alpha)x_{t+1}] \hat{z}_{t+1} \). Similarly, we define \( z_{t+1} \equiv \beta \exp [(1 - \alpha)x_{t+1}] (y_{t+1} + 1) \) and use this definitional relationship to eliminate \( y_{t+1} + 1 \). In
the expression for \( R^*_{t+1} \), we have substituted in \( (y^*_{t+1} + 1) / y^*_t = \beta^{-1} \exp [-(1 - \alpha)x_{t+1}] \) from
the nonlinear law of motion (17). Note that the three return measures differ only by the
terms \( \hat{z}_{t+1}/E(\hat{z}_{t+1}|H_t) \) and \( z_{t+1}/E(z_{t+1}|I_t) \), which are measures of the investor’s proportional
forecast errors under the different information assumptions.

In Appendix A.2, we show that the approximate law of motion for \( \log(\hat{R}_{t+1}) \) is

\[
\log(\hat{R}_{t+1}) - E[\log(\hat{R}_{t+1})] = \alpha (x_{t+1} - \bar{x}) + a_1 z_{t+1} \quad (30)
\]

where \( a_1 \) is given by Proposition 1.

In Appendix B.2, we show that when \( I_t = H_t \cup d_{t+1} \), the approximate law of motion for
\( \log(R_{t+1}) \) is

\[
\log(R_{t+1}) - E[\log(R_{t+1})] = n_1 (x_{t+2} - \bar{x}) + (1 - a_1) (x_{t+1} - \bar{x}), \quad (31)
\]

where \( n_1 = a_1 \exp (a_0) / [1 + \exp (a_0)] \) is a Taylor-series coefficient.

In Appendix C.2, we show that the exact law of motion for \( \log(R^*_{t+1}) \) is

\[
\log \left( R^*_{t+1} \right) - E \left[ \log \left( R^*_{t+1} \right) \right] = \alpha (x_{t+1} - \bar{x}). \quad (32)
\]
Given the above laws of motion, it is straightforward to compute the following unconditional variances:

\[
\begin{align*}
\text{Var}[\log(\hat{R}_{t+1})] &= \alpha^2 \text{Var}(x_t) + a_1 [a_1 + 2\alpha] \sigma^2, \\
\text{Var}[\log(R_{t+1})] &= \left[(n_1)^2 + (1 - a_1)^2 + 2n_1 (1 - a_1) \rho\right] \text{Var}(x_t). \\
\text{Var}[\log(R_{t+1}^*)] &= \alpha^2 \text{Var}(x_t).
\end{align*}
\]

4.1 Results for Special Cases

LeRoy and Parke (1992) considered the special case of risk neutrality and iid dividend growth. In the present setting, under \(\alpha = \rho = 0\), Proposition 1 implies \(a_1 = 1\) and equation (13) implies \(\text{Var}(x_t) = \sigma^2\). The variance expressions imply the following inequality:

\[
\begin{align*}
\text{Var}[\log(R_{t+1}^*)] &\leq \text{Var}[\log(R_{t+1})] \leq \text{Var}[\log(\hat{R}_{t+1})], \quad \text{when } \alpha = \rho = 0, \\
&= \text{Var}(x_t) = \text{Var}[\log(R_{t+1})] = \text{Var}[\log(\hat{R}_{t+1})], \quad \text{when } \rho = 0,
\end{align*}
\]

where \(a_1 = 1\) implies \(n_1 = \exp(a_0) / [1 + \exp(a_0)] < 1\). In this example where \(I_t = H_t \cup d_{t+1}\), the variance of the perfect foresight return represents a lower bound of zero while the variance of the return based on information set \(H_t\) represents an upper bound. This finding agrees with that of LeRoy and Parke (1992) in a similar (but not identical) setting.

However, it is straightforward to show by counterexample that similar bounds do not extend to the case where investors are risk averse. Consider the following counterexample when \(\rho = 0\) but \(\alpha \neq 0\). We have

\[
\begin{align*}
\text{Var}[\log(R_{t+1}^*)] &\leq \text{Var}[\log(R_{t+1})] \leq \text{Var}[\log(\hat{R}_{t+1})], \quad \text{when } \rho = 0, \\
&= \text{Var}(x_t) = \text{Var}[\log(R_{t+1})] = \text{Var}[\log(\hat{R}_{t+1})]
\end{align*}
\]

where the direction of the second inequality depends on the magnitude of \(\alpha\) and \(n_1\). Starting from information set \(H_t\) corresponding to \(\hat{R}_{t+1}\), an increase in investor information can either increase or decrease the log return variance, depending on parameter values.

In the special case of log utility, we have \(\alpha = 1\) such that \(a_1 = n_1 = 0\). This case is not a counterexample because it implies

\[
\begin{align*}
\text{Var}[\log(R_{t+1}^*)] = \text{Var}[\log(R_{t+1})] = \text{Var}[\log(\hat{R}_{t+1})]
\end{align*}
\]

for every specification of \(I_t\). Since the price-dividend ratio is constant under log utility regardless of investors’ information, return variance is driven solely by the exogenous stochastic process for dividends.
From equations (33) and (35), equality of $\text{Var}[\log(\hat{R}_{t+1})]$ and $\text{Var}[\log(R_{t+1}^*)]$ can also occur when $a_1 + 2\alpha = 0$. We will verify below that at least for some parameter specifications there exists a positive value of $\alpha$ that satisfies this equation. The critical value of $\alpha$ defines a crossing point: at this value of $\alpha$ the size ordering between $\text{Var}[\log(\hat{R}_{t+1})]$ and $\text{Var}[\log(R_{t+1}^*)]$ reverses. Further, it turns out that at the critical value of $\alpha$ we have $\text{Var}[\log(\hat{R}_{t+1})] > \text{Var}[\log(R_{t+1}^*)]$, where $\text{Var}[\log(R_{t+1}^*)]$ is the return variance based on $I_t = H_t \cup d_{t+1}$ We therefore conclude that $\text{Var}[\log(\hat{R}_{t+1})]$ and $\text{Var}[\log(R_{t+1}^*)]$ cannot be bounds for return volatility. This result should not be surprising. Unlike the situation with prices, the returns that prevail under either information set $H_t$ and $I_t$ cannot be represented as conditional forecasts of the return that prevails under perfect foresight.

4.2 Quantitative Analysis

Solving for the critical value of $\alpha$ where $\text{Var}[\log(\hat{R}_{t+1})] = \text{Var}[\log(R_{t+1}^*)]$ can be accomplished analytically using the following approximate expression for the solution coefficient $a_1$ in Proposition 1: $a_1 \simeq (1-\alpha) / (1-\rho\beta)$. The approximation expression for $a_1$ is derived by assuming $\exp \left[ (1-\alpha) \pi + (a_1)^2 \sigma_{\varepsilon}^2 / 2 \right] \simeq 1$ which holds exactly when $\alpha = 1$ and remains reasonably accurate for $\alpha < 10$. Substituting the approximate expression for $a_1$ into the variance equality condition $a_1 + 2\alpha = 0$ and then solving for $\alpha$ yields a second value for $\alpha$ for which $\text{Var}[\log(\hat{R}_{t+1})] = \text{Var}[\log(R_{t+1}^*)]$. This second value is $\alpha \simeq 1 / (2\rho\beta - 1)$. Positivity of $\alpha$ requires that the model parameters satisfy $\rho\beta > 0.5$. This counterexample establishes that $\text{Var}[\log(R_{t+1}^*)]$ cannot be a lower bound because it may be greater than or less than $\text{Var}[\log(\hat{R}_{t+1})]$ depending on the value of $\alpha$.

Figure 2 plots the volatilities of log returns for two different calibrations of the model. In the top panel, we employ the same calibration as Figure 1 with $\rho = -0.1$ to match the autocorrelation of U.S. consumption growth from 1890 to 2004. In the bottom panel, we set $\rho = 0.7$ and recalibrate the value of $\sigma_{\varepsilon}$ to maintain the same standard deviation of consumption growth as in the top panel.

In the top panel, we see that the volatility of $\log(\hat{R}_{t+1})$ is equal to the volatility of $\log(R_{t+1}^*)$ only when $\alpha = 1$. In the bottom panel, the model parameters satisfy $\rho\beta > 0.5$ so the two return volatilities are also equal when $\alpha \simeq 1 / (2\rho\beta - 1) = 2.9$. As $\alpha$ crosses the values 1 and 2.9, the direction of the variance inequality comparing the volatility of $\log(\hat{R}_{t+1})$ to that of $\log(R_{t+1}^*)$ reverses direction. As observed above, such reversals demonstrate that $\text{Var}[\log(R_{t+1}^*)]$ cannot be a lower bound for all $\alpha$.\textsuperscript{18}

\textsuperscript{18}Lansing (2010) provides a counterexample to the variance bounds derived by Engel (2005) involving log price changes under the assumption of risk neutrality. He shows that the direction of the variance inequality depends on the value of the dividend persistence parameter.
5 Conclusion

We see that for low levels of risk aversion, the volatility of actual price-dividend ratios in U.S. data greatly exceed their upper bounds under the present-value model. Thus we reproduce the result found in the earlier variance-bounds literature. However, the finding of excess volatility disappears under the assumption that investors have high risk aversion: for $\alpha > 5$ we find that the volatility of the actual price-dividend ratio is reasonably close to the volatility of the perfect foresight price-dividend ratio. Thus the volatility of stock prices is about as one would expect under the assumption that investors can forecast future dividends accurately into the distant future.

To be sure, the assumption that investors have such foresight may be viewed as implausible. If one includes in the null hypothesis the assumption that investors can forecast dividend growth at most only a few years in the future, as may be thought more realistic, then actual stock price volatility appears excessive for any level of risk-aversion.\footnote{Mehra and Prescott (1985) argue that risk aversion coefficients below 10 are plausible.}

As regards to return volatility, we show that the bounds on excess payoff variance derived under the assumptions of risk neutrality and $iid$ dividend growth do not extend to bounds on returns under risk aversion, except in some special cases.

As LeRoy and Parke (1992) noted, a case can be made that the geometric random walk specification ($\rho = 0$) is not a bad approximation; we found that the data favor $\rho \approx -0.1$. However, the conclusion here is that the assumption of risk neutrality adopted in the variance-bounds literature, including LeRoy and Parke (1992), is indeed very restrictive: if risk aversion is low, then price volatility in the data is clearly higher than is justified by dividend volatility. However, if one takes the view that risk aversion coefficients above 5 are admissible, then price volatility in the data may be consistent with the present-value model. Thus ruling out risk aversion, as was done in earlier variance bounds tests, has seriously adverse implications to the extent that investors are risk averse.

This conclusion is reminiscent of that of Mehra and Prescott (1985), who found that they could not explain the equity premium in models that assumed low or moderate levels of risk aversion, but could do so with extremely high levels of risk aversion. The difference in their analysis was that the levels of risk aversion needed to explain the equity premium in the data were on the order of 50, whereas here much lower levels of risk aversion will serve the purpose of explaining the volatility of the price-dividend ratio in the data.
A Solution under Information Set $H_t$

A.1 Proof of Proposition 1

Iterating ahead the law of motion for $\tilde{z}_t$ specified in Proposition 1 and taking the conditional expectation implied by the information set $H_t$ yields

$$E(\tilde{z}_{t+1}|H_t) = \tilde{y}_t = \exp \left[ a_0 + \rho a_1 (x_t - \bar{x}) + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon \right]. \quad (A.1)$$

Substituting the above expression into the first order condition (19) and then taking logarithms yields

$$\log (\tilde{z}_t) = F(x_t) = \log (\beta) + (1 - \alpha)x_t + \log \left\{ \exp \left[ a_0 + \rho a_1 (x_t - \bar{x}) + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon \right] + 1 \right\} \simeq a_0 + a_1 (x_t - \bar{x}), \quad (A.2)$$

where the Taylor-series coefficients $a_0$ and $a_1$ are given by

$$a_0 = F(\bar{x}) = \log (\beta) + (1 - \alpha) \bar{x} + \log \left\{ \exp \left[ a_0 + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon \right] + 1 \right\}, \quad (A.3)$$

$$a_1 = \frac{\partial F}{\partial x_t} \bigg|_{\bar{x}} = 1 - \alpha + \frac{\rho a_1 \exp \left[ a_0 + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon \right]}{\exp \left[ a_0 + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon \right] + 1}. \quad (A.4)$$

Solving equation (A.3) for the unconditional mean $a_0$ yields

$$a_0 = E[\log(\tilde{z}_t)] = \log \left\{ \frac{\beta \exp \left[ (1 - \alpha) \bar{x} \right]}{1 - \beta \exp \left[ (1 - \alpha) \bar{x} + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon \right]} \right\}, \quad (A.5)$$

which can be substituted into equation (A.4) to yield the following nonlinear equation that determines $a_1$:

$$a_1 = 1 - \alpha + \rho a_1 \beta \exp \left[ (1 - \alpha) \bar{x} + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon \right]. \quad (A.6)$$

Rearranging equation (A.6) yields the expression shown in Proposition 1. There are two solutions, but only one solution satisfies the condition $\beta \exp \left[ (1 - \alpha) \bar{x} + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon \right] < 1$, which is verified after solving (A.6) via a nonlinear equation solver.

A.2 Asset Pricing Moments

This section briefly outlines the derivation of equations (21) and (33). Taking the unconditional expectation of $\log(\tilde{y}_t)$ in equation (20) yields

$$E[\log(\tilde{y}_t)] = a_0 + \frac{1}{2} (a_1)^2 \sigma^2_\varepsilon. \quad (A.7)$$
We then have
\[ \log \left( \widehat{y}_t \right) - E \left[ \log \left( \widehat{y}_t \right) \right] = a_1 \rho \left( x_t - \overline{x} \right), \] (A.8)
which in turn implies
\[ \text{Var} \left[ \log \left( \widehat{y}_t \right) \right] = (a_1 \rho)^2 \text{Var} \left( x_t \right). \] (A.9)

As described on the text, the equity return \((27)\) implied by the information set \(H_t\) can be rewritten as
\[ \frac{\widehat{R}_{t+1}}{\beta} = \exp \left( \alpha x_{t+1} + \frac{\widehat{z}_{t+1}}{E(z_{t+1}|H_t)} \right). \] (A.10)

Substituting in \(E(z_{t+1}|H_t)\) from equation (A.1) and \(\widehat{z}_{t+1} = \exp \left[ a_0 + a_1 \left( x_{t+1} - \overline{x} \right) \right]\) from Proposition 1 and then taking the unconditional mean of \(\log(\widehat{R}_{t+1})\) yields
\[ E \left[ \log(\widehat{R}_{t+1}) \right] = -\log(\beta) + \alpha \overline{x} - \frac{1}{2} \left( a_1 \right)^2 \sigma_x^2. \] (A.11)

We then have
\[ \log(\widehat{R}_{t+1}) - E \left[ \log(\widehat{R}_{t+1}) \right] = \alpha \left( x_{t+1} - \overline{x} \right) + a_1 \varepsilon_{t+1}, \] (A.12)
which in turns implies
\[ \text{Var} \left[ \log(\widehat{R}_{t+1}) \right] = \alpha^2 \text{Var} \left( x_t \right) + \left( a_1 \right)^2 \sigma_x^2 + 2 \alpha a_1 \sigma_x^2 + 2 \alpha \sigma_x^2 \left( \text{Cov} \left( x_{t+1}, \varepsilon_{t+1} \right) \right). \] (A.13)

**B Solution under Information Set \(I_t = H_t \cup d_{t+1}\)**

Iterating ahead the first-order condition (8) and then imposing the equilibrium relationship \(c_t = d_t\) for all \(t\) yields
\[ p_t = \beta \left( \frac{d_{t+1}}{d_t} \right)^{-\alpha} \left( d_{t+1} + \widehat{p}_{t+1} \right), \] (B.1)
where \(\widehat{p}_{t+1}\) is the equilibrium price conditional on the information set \(H_{t+1}\).

Dividing both sides of equation (B.1) by \(d_t\) yields the following expression for \(y_t \equiv p_t/d_t:\)
\[ y_t = \beta \exp \left[ (1 - \alpha) x_{t+1} \right] \left[ 1 + \widehat{y}_{t+1} \right], \]
\[ = \beta \exp \left[ (1 - \alpha) x_{t+1} \right] \left[ 1 + E(\widehat{z}_{t+2}|H_{t+1}) \right], \]
\[ = \widehat{z}_{t+1}, \] (B.2)
where we have made use of the definition \(\widehat{y}_{t+1} \equiv \widehat{p}_{t+1}/d_{t+1}\) and the equilibrium relationship (19).

Given that \(y_t = \widehat{z}_{t+1}\) from equation (B.2) and \(\widehat{y}_t = E(\widehat{z}_{t+1}|H_t)\) from equation (A.1), we then have \(\widehat{y}_t = E(y_t|H_t)\) which implies \(\text{Var} \left( \widehat{y}_t \right) \leq \text{Var} \left( y_t \right)\).
B.1 Asset Pricing Moments

This section outlines the derivation of equations (24) and (34). From equations (A.2) and (B.2) we have the following approximate law of motion for $y_t$

$$y_t = \tilde{z}_{t+1} \simeq \exp \left[ a_0 + a_1 (x_{t+1} - \bar{x}) \right]. \quad (B.3)$$

The above expression implies

$$\text{Var} \left[ \log (y_t) \right] = (a_1)^2 \text{Var} (x_t). \quad (B.4)$$

The equity return (28) implied by the information set $I_t$ can be rewritten as

$$R_{t+1} = \exp (x_{t+1}) \left[ \frac{\tilde{z}_{t+2} + 1}{\tilde{z}_{t+1}} \right], \quad (B.5)$$

where we have eliminated both $y_t$ and $y_{t+1}$ using equation (B.2). The approximate law of motion for $\tilde{z}_{t+1}$ is given by equation (B.3). An approximate law of motion for $\tilde{z}_{t+2} + 1$ is given by

$$\tilde{z}_{t+2} + 1 \simeq \exp \left[ n_0 + n_1 (x_{t+2} - \bar{x}) \right], \quad (B.6)$$

where $n_0 = 1 + \exp (a_0)$ and $n_1 = a_1 \exp (a_0) / \left[ 1 + \exp (a_0) \right]$ are Taylor series coefficients.

Substituting equations (B.3) and (B.6) into (B.5) and then taking the unconditional mean of $\log (R_{t+1})$ yields

$$E[\log (R_{t+1})] = n_0 - a_0 + \bar{x}. \quad (B.7)$$

We then have

$$\log (R_{t+1}) - E[\log (R_{t+1})] = n_1 (x_{t+2} - \bar{x}) + (1 - a_1) (x_{t+1} - \bar{x}). \quad (B.8)$$

Squaring both sides of equation (B.8) and taking the unconditional mean yields the expression for $\text{Var} [\log (R_{t+1})]$ shown in equation (34).

C Solution under Perfect Foresight

C.1 Log-linearized Law of Motion

Taking logarithms of the nonlinear law of motion (17) yields

$$\log (y^*_t) = G \left[ x_{t+1}, \log (y^*_{t+1}) \right] = \log (\beta) + (1 - \alpha) x_{t+1} + \log \left\{ \exp \left[ \log (y^*_{t+1}) \right] + 1 \right\}$$

$$\simeq b_0 + b_1 (x_{t+1} - \bar{x}) + b_2 \left[ \log (y^*_{t+1}) - b_0 \right], \quad (C.1)$$
where the Taylor-series coefficients $b_0$, $b_1$ and $b_2$ are given by

\begin{align*}
    b_0 &= G(\bar{x}, b_0) = \log(\beta) + (1 - \alpha) \bar{x} + \log[\exp(b_0) + 1], \quad (C.2) \\
    b_1 &= \left. \frac{\partial G}{\partial x_t} \right|_{\bar{x}, b_0} = 1 - \alpha, \quad (C.3) \\
    b_2 &= \left. \frac{\partial G}{\partial \log(y_{t+1})} \right|_{\bar{x}, b_0} = \frac{\exp(b_0)}{\exp(b_0) + 1}. \quad (C.4)
\end{align*}

Solving equation (C.2) for the unconditional mean $b_0$ yields

\[ b_0 = E[\log(y_t^*)] = \log\left\{ \frac{\beta \exp[(1 - \alpha) \bar{x}]}{1 - \beta \exp[(1 - \alpha) \bar{x}]} \right\}, \quad (C.5) \]

which can be substituted into equation (C.4) to obtain the following expression for $b_2$:

\[ b_2 = \beta \exp[(1 - \alpha) \bar{x}]. \quad (C.6) \]

Subtracting the unconditional mean $b_0 = E[\log(y_t^*)]$ from both sides of the approximate law of motion (C.1) and then substituting for $b_1$ and $b_2$ from (C.3) and (C.6) yields equation (25).

### C.2 Asset Pricing Moments

This section outlines the derivation of equations (26) and (35). Squaring both sides of equation (25) and then taking the unconditional mean to obtain the variance yields

\[ \text{Var}[\log(y_t^*)] = \frac{(1 - \alpha)^2 \text{Var}(x_t) + 2(1 - \alpha) \beta \exp[(1 - \alpha) \bar{x}] \text{Cov}[\log(y_t^*), x_t]}{1 - \beta^2 \exp[2(1 - \alpha) \bar{x}]].} \quad (C.7) \]

The next step is to compute $\text{Cov}[\log(y_t^*), x_t]$ which appears in equation (C.7). Starting from equation (25), we have

\[ \text{Cov}[\log(y_t^*), x_t] = (1 - \alpha) \text{Cov}(x_{t+1}, x_t) = \rho \text{Cov}(x_t, x_{t-1}) + \beta \exp[(1 - \alpha) \bar{x}] \text{Cov}[\log(y_{t+1}^*), x_t], \quad (C.8) \]

\[ \text{Cov}[\log(y_{t+1}^*), x_t] = (1 - \alpha) \rho \text{Cov}(x_t, x_{t-1}) + \beta \exp[(1 - \alpha) \bar{x}] \text{Cov}[\log(y_{t+2}^*), x_t], \quad (C.9) \]
and so on for $\text{Cov} \left[ \log (y_{t+j}^*), x_t \right], j = 1, 2, \ldots$. By repeated substitution to eliminate $\text{Cov} \left[ \log (y_{t+j}^*), x_t \right]$ and then applying a transversality condition, we obtain the following expression:

$$\text{Cov} \left[ \log (y_t^*), x_t \right] = (1 - \alpha) \text{Cov} (x_t, x_{t-1}) \sum_{j=0}^{\infty} \rho \beta \exp \left[(1 - \alpha) \overline{\pi} \right]j$$

$$= \frac{(1 - \alpha) \text{Cov} (x_t, x_{t-1})}{1 - \rho \beta \exp \left[(1 - \alpha) \overline{\pi} \right]}, \quad (C.10)$$

where the infinite sum converges provided that $\rho \beta \exp \left[(1 - \alpha) \overline{\pi} \right] < 1$. Substituting equation (C.10) into equation (C.7), together with $\text{Cov} (x_t, x_{t-1}) = \rho \text{Var} (x_t)$ and then simplifying yields equation (26).

The perfect foresight return (29) can be rewritten as

$$R_{t+1}^* = \beta^{-1} \exp (\alpha x_{t+1}), \quad (C.11)$$

where we have substituted in $(y_{t+1}^* + 1)/y_t^* = \beta^{-1} \exp \left[-(1 - \alpha) x_{t+1} \right]$ from the exact nonlinear law of motion (17). Taking the unconditional expectation of $\log (R_{t+1}^*)$ yields

$$E \left[ \log (R_{t+1}^*) \right] = - \log (\beta) + \alpha \overline{x} \quad (C.12)$$

We then have

$$\log (R_{t+1}^*) - E \left[ \log (R_{t+1}^*) \right] = \alpha (x_{t+1} - \overline{x}), \quad (C.13)$$

which in turns implies the unconditional variance (35).
References


Figure 1: The log price-dividend ratio in U.S. data exhibits excess volatility for $\alpha < 5$. 
Figure 2: The present-value model does not impose bounds on returns in general settings involving risk aversion.