

Present Value

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The present-value relation says that, under certainty, the value of a capital good or financial asset equals the summed discounted value of the stream of revenues which that asset generates. The discount factor is that determined by the interest rate over the relevant period. The justification for the present-value relation lies in the fact that (in perfect capital markets) an asset must earn a rate of return exactly equal to the interest rate. Otherwise arbitrage opportunities emerge, which is inconsistent with equilibrium. Thus if r_t is the one-period interest rate at t , p_t is the (ex-dividend) price of an asset and d_t is its dividend, it must be true that

$$r_t = (d_{t+1} + p_{t+1})/p_t - 1, \quad (1)$$

since the right-hand side equals the rate of return on the asset. Solving for p_t ,

$$p_t = \frac{d_{t+1} + p_{t+1}}{1 + r_t}. \quad (2)$$

Replacing t by $t + 1$, (2) becomes an equation expressing p_{t+1} as a function of r_{t+1} , d_{t+1} and p_{t+2} . Substituting this in (2) and proceeding similarly n times, it follows that

$$p_t = \sum_{i=1}^n \frac{d_{t+i}}{\prod_{j=0}^{i-1} (1 + r_{t+j})} + \frac{p_{t+n}}{\prod_{j=0}^{n-1} (1 + r_{t+j})}. \quad (3)$$

Assuming that speculative price bubbles do not occur (see below), the right-most term in (3) converges to zero as n goes to infinity, so there results the present value equation

$$p_t = \sum_{i=1}^{\infty} \frac{d_{t+i}}{\prod_{j=0}^{i-1} (1 + r_{t+j})}. \quad (4)$$

If in addition the interest rate is constant at $r_t = \rho$, (3) may be written as

$$p_t = \sum_{i=1}^n (1 + \rho)^{-i} d_{t+i} + (1 + \rho)^{-n} p_{t+n}. \quad (5)$$

or, if the convergence condition is satisfied, as

$$p_t = \sum_{i=1}^{\infty} (1 + \rho)^{-i} d_{t+i}. \quad (6)$$

In the special cases in which d_{t+i} is constant at d , or grows from d_t at rate g , (6) simplifies to

$$p_t = \frac{d}{\rho} \quad (7)$$

or

$$p_t = \frac{d_t(1 + g)}{\rho - g}, \quad (8)$$

respectively.

In introductory finance courses, the present-value relation makes an early appearance in the chapter on capital budgeting, where it is taught that corporations should accept any investment project that promises a positive present value (net of costs), and only these. This wealth-maximization decision rule is the correct one independent of agents' preferences because, regardless of preferences, the consumption set that it generates dominates that generated by any other capital budgeting criterion. This is Fisher's separation principle. Other criteria, such as accepting that project with the shortest payback period, or that with the highest internal rate of return, are either equivalent to present-value maximization, ambiguous (sometimes, for example, a single project may have no real internal rate of return, or more than one) or wrong, depending on the characteristics of the project's returns.

Under uncertainty, but assuming risk neutrality, the present-value relation may be written as

$$p_t = \sum_{i=1}^{\infty} (1 + \rho)^{-i} E_t(d_{t+i}), \quad (9)$$

which differs from (6) only in that future dividends is replaced by their conditional expectation. This version of the present-value relation has received extensive study, especially in the early finance literature. It is easily shown to imply

$$E_t(r_t) = \rho, \quad (10)$$

saying that the conditional expectation of the rate of return on the asset equals a constant independent of the conditioning set (Samuelson [11], [12]). This strong

restriction provides the basis for most empirical tests of what has been called ‘capital market efficiency’ (Fama [1] ; LeRoy, [7]): if (10) is true, no information publicly available at t should be correlated with the rate of return on the asset from t to $t + 1$. In this sense prices ‘fully reflect’ all publicly available information.

The present-value relation may also be interpreted from the vantage point of its martingale implication: if the asset is priced according to (9), the value x_t of a mutual fund which holds the asset and reinvests all of its dividend income will follow a martingale with drift defined by

$$E_t(x_{t+1}) = (1 + \rho)x_t. \quad (11)$$

To see this, assume that the mutual fund holds h_t shares of the asset so that

$$x_t = h_t p_t \quad \text{and} \quad x_{t+1} = h_{t+1} p_{t+1}. \quad (12)$$

When dividend income is reinvested, h_{t+1} is given by

$$h_{t+1} p_{t+1} = h_t (p_{t+1} + d_{t+1}). \quad (13)$$

Then

$$E_t(x_{t+1}) = E_t(h_{t+1} p_{t+1}) = h_t E_t(d_{t+1} + p_{t+1}) = x_t (1 + \rho), \quad (14)$$

using (1) and (10). To see that the correction for dividends payout is needed, observe that (10) implies that

$$\rho = \frac{E_t(d_{t+1})}{p_t} + \frac{E_t(p_{t+1})}{p_t} - 1, \quad (15)$$

so that changes in the expected dividend yield are always offset one-for-one by changes in the expected rate of capital gain. If p_t by itself were a martingale the expected rate of capital gain would be a constant, implying that p_t is a constant multiple of expected dividends. But this is not an implication of the present-value relation (take dividends as given by a first-order autoregressive process, for example). Hence p_t by itself does not follow a martingale.

The present-value-martingale model appears in many contexts in finance. If a futures price is assumed equal to the conditional expectation of the relevant spot price, then the futures price will follow a martingale (Samuelson [11]). If owners of an exhaustible resource like petroleum extract it at optimal rates, then in some settings the price of reserves will appreciate according to a martingale with drift equal to the interest rate. Finally, the expected present-value relation has implications for the volatility of asset prices. Informally, the expected present-value relation implies that stock prices are like a moving average of the dividend stream to which they give title. Since a moving average is smoother than its components, it follows that

stock prices should show less volatility than dividends. Volatility tests along these lines were originally reported by Shiller [14] and LeRoy and Porter [9]. A number of subsequent papers extended and criticized the finding of excess volatility.

Equation (10), which requires that the conditional expectation of the rate of return does not depend on the value taken on by the conditioning variables, is very restrictive. Unlike its certainty analogue (1), which reflects only the assumption of zero transactions costs, (10) constitutes a restriction on the equilibrium probability distribution of the endogenously determined stock prices much stronger than anything implied by the idea of capital market efficiency alone. The question becomes: what restrictions on preferences and the production technology are needed to derive (10)? LeRoy [5] showed that if agents are risk-neutral, then in an exchange economy (10) will be satisfied (see also LeRoy [6] for discussion in a more general setting). The result is a consequence of the obvious fact that if agents are risk neutral they will ignore moments in the distribution of rates of return higher than the first. Under nonzero risk aversion, however, the conditional expected rate of return will contain a risk premium which generally depends on the realizations of the conditioning variables. Hence (10) will generally not be true. LeRoy [5] and Lucas [10] discussed a class of models in which the expected present-value property fails except as a special case.

If the assumption of risk neutrality is relaxed the valuation equations must be changed. This can be achieved either by modifying the characterization of expected cash flows or by respecifying the discount factor. Modifying the characterization of expected cash flows involves distorting (relative to natural probabilities) the probability measure used to take expectations so as to put greater (lesser) weight on states in which agents have high (low) marginal utility. Such distorted probabilities always exist in the finite case, and exist under weak assumptions generally. When these “risk-neutral probabilities” are used to compute expectations, security prices equal expected payoffs discounted at the interest rate, just as under risk neutrality (hence the name).

Alternatively, one can retain the natural probabilities, but adjust the discount factor to allow for risk aversion. Under the Capital Asset Pricing Model the risk premium on any security or portfolio is proportional to a beta coefficient, which equals the regression coefficient of the return on the return on the market portfolio. The constant of proportionality is the risk premium on the market portfolio. The idea is that risk-averse agents require high expected returns on high-beta securities since such securities increase portfolio risk on the margin.

Returning to the certainty case, if the rate of return on an asset is constant at ρ but the convergence condition

$$\lim_{n \rightarrow \infty} (1 + \rho)^{-n} p_{t+n} = 0 \tag{16}$$

is not satisfied, then asset prices are characterized by a speculative bubble. For a nontechnical introduction to rational bubbles, see LeRoy [8]. The asset’s price is higher than the present value of the stream of dividends the asset is title to, but

nonetheless investors are willing to hold the asset because its price is expected to rise in the future. The definition of speculative bubbles under uncertainty is analogous (whether speculative bubbles exist or not has nothing directly to do with uncertainty). If speculative bubbles can occur, the present value equation must be generalized to

$$p_t = \sum_{i=1}^{\infty} (1 + \rho)^{-i} d_{t+i} + \gamma(1 + \rho)^t, \quad (17)$$

where γ is an arbitrary non-negative constant capturing the magnitude of the speculative bubble. Equation (17) is the class of solutions to the difference equation

$$\rho = \frac{(d_{t+1} + p_{t+1})}{p_t} - 1, \quad (18)$$

where γ is the constant of integration ($\gamma \geq 0$ results from the requirement that asset prices be always non-negative, a consequence of free disposal). In the special case $\gamma = 0$ speculative bubbles are absent and the present value relation results.

Bubbles cannot occur on a security that has a payoff only at one date, such as a zero-coupon bond. By induction, the same is true of securities that have payoffs at a finite number of dates. Existence of a bubble on such assets would imply an arbitrage opportunity: investors could sell the security short and purchase claims for its payoff at a cost equal to the present value of those payoffs. In the case of securities with payoffs at an infinite number of dates, it may or may not be possible to rule out bubbles on theoretical grounds. A few of the many papers dealing with this question are Tirole [15], Gilles and LeRoy [2], [3], Santos and Woodford [13] and Huang and Werner [4].

See also finance; internal rate of return; investment decision criteria; martingales, excess volatility.

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