

Infinite Portfolio Strategies*

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Abstract

Continuous-time stochastic calculus extends the definition of the stochastic integral to the case where the integrand is not square-integrable using a pointwise limit. Under this extension absence of arbitrage in finite portfolio strategies is consistent with existence of arbitrage in infinite portfolio strategies. The doubling strategy is the most common example. We argue that this extension does not make economic sense, and propose an alternative extension of the definition of the stochastic integral under which absence of arbitrage is preserved. The extension involves appending a date and state called ∞ to the payoff index set, and defining a topology such that the set of payoffs is compact. Doing so allows a simplified mathematical treatment of a number of topics that are unwieldy when modeled in a setting where the payoff index set is noncompact. Applications discussed include Ponzi schemes and payoff bubbles.

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In continuous-time finance theory, portfolio payoffs are defined using the Ito definition of the stochastic integral. Security prices x are modeled as a martingale—most simply, as a Brownian motion—and a portfolio strategy is represented by a predictable stochastic process θ . Assuming that portfolio strategies obey

$$\int_{t=0}^T E(\theta_t^2) dt < \infty \quad (1)$$

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the definition of the integral implies that for any $\tau \leq T$ the value at time τ of the portfolio, given by

$$\int_{t=0}^{\tau} \theta_t dx_t, \tag{2}$$

is a random variable with finite mean and variance.

In many expositions it is pointed out that the definition of the Ito stochastic integral, and with it the characterization of payoffs of portfolio strategies, can be extended to cases in which the condition (1) is not satisfied. To see this, suppose that (1) fails, but the weaker condition

$$\int_{t=0}^{\tau} E(\theta_t^2) dt < \infty \tag{3}$$

is satisfied for all $\tau < T$. In that case the integral

$$\int_{t=0}^{\tau} \theta_t dx_t \tag{4}$$

can be defined for each $\tau < T$, and therefore the integral (2) can be defined as the pointwise limit of (4) as τ approaches T (a similar extension of the definition is available when x is assumed to be a local martingale rather than a martingale; we will not discuss this case).

Neither the mathematics-oriented sources for this extension (for example, Chung and Williams [3]) nor the finance-oriented sources (for example, Duffie [5], Appendix D) provide much in the way of motivation for identifying the extension with a pointwise limit. Inasmuch as the Ito integral is based on the stochastic process for x itself and not on the sample paths of x (which, as is well known, in the case of Brownian motion are too choppy to be integrated), it appears that the extended definition of the integral has little in common with the original definition. The reader is left uncertain whether the extension of the integral involves economic assumptions beyond those involved in the standard case. We look into this question in the present paper, finding that the extension is indeed questionable in a sense that the original integral (2) is not.

The extension just defined turns out to imply a discontinuity in the valuation of portfolio payoffs and, under weak assumptions, the utilities that these payoffs generate: the values of payoffs of portfolio strategies for $\tau < T$ do not converge to those of portfolio strategies at T . In other words, valuation is pointwise discontinuous. We take the view that there is no economic motivation for this discontinuity, and its existence raises the question whether the pointwise definition of convergence is appropriate.

It should be emphasized that we do not claim that there is a mathematical problem with the extension of the definition of the stochastic integral. We do suggest that the extension is not suitable for finance applications; specifically, that it does not provide an economically sensible characterization of portfolio payoffs. The situation is similar to the comparison often made between the Ito integral and the Stratonovich integral in favor of the former:¹ there is nothing wrong with the Stratonovich integral, but for economic reasons it is the wrong choice in the context of finance.

If our argument for the unsuitability of the extension of the Ito integral is accepted, there are two possible ways to proceed. One is to revoke the status as bona fide portfolio strategies of candidate portfolio strategies that do not satisfy the condition (1), even when they do satisfy the weaker restriction (3). Thus we simply assume that investors do not choose those candidate portfolio strategies for which (1) is not satisfied. The other way to proceed is to adopt a different and more sensible definition of portfolio payoffs when the condition (1) fails. We will set out below a definition of the extension of the stochastic integral that meets this standard.

The revised definition is of independent interest. It turns out that portfolio strategies for which the condition (1) fails are exactly those that have nonzero bubble components. The revised definition of portfolio payoffs is seen to allow a treatment of bubbles that is much simpler mathematically than that available in the bubbles literature.

In order to simplify the exposition we adopt a discrete-time setting. In our chosen setting portfolio strategies that are terminated at date $\tau < T$ involve a finite number of transactions, so in that case there is no possible ambiguity about how to define portfolio payoffs. In contrast, portfolio strategies identified with terminal date T involve an infinite number of transactions. In this setting the task becomes that of determining the consequences of defining payoffs of infinite portfolio strategies as the pointwise limits of payoffs of finite portfolio strategies.

1 The Doubling Strategy

The prototype candidate portfolio strategy for which the condition (1) fails is the doubling strategy, under which the investor doubles the investment in a risky asset until he generates a gain, and then stops. The doubling strategy is discussed in practically every introductory graduate-level text in financial economics, but most discussions fail to make the connection with the stochastic integral (for example,

¹It will be recalled that in the Ito integral the integrand is assumed to be predictable, so that in the discrete version the portfolio weights are those associated with the leftmost point of each interval. In contrast, under the Stratonovich integral the value of the integrand is instead taken at the midpoint of the discrete intervals. The Stratonovich integral is unsuited for the definition of portfolio payoffs because, obviously, investors cannot base portfolio choices for any period on security prices prevailing half-way through the period; they have to use the prices prevailing at the beginning of the period.

Etheridge [6]).

Because the portfolio weights under the doubling strategy do not satisfy (1), the payoff of the doubling strategy is defined as the pointwise limit of the payoffs of the truncated doubling strategies. This limit envisions the investor gaining \$1 with probability 1, an arbitrage. In other words, a portfolio strategy that is increasingly unattractive to risk-averse investors when terminated in finite time morphs into a riskless arbitrage in infinite time. Such arbitrages, it is held, must be ruled out by restrictions on admissible portfolio strategies.

This specification does not correspond in any way to how actual investors behave. There is no reason to introduce such discontinuities in valuation in analyses that are intended to have empirical relevance. Some analysts, to their credit, appear to be uneasy about labeling the doubling strategy an arbitrage, particularly in the absence of any characterization of preferences. For example, Delbaen and Schachermayer [4] concluded from the fact that losses are unbounded under the doubling strategy that “[e]verybody, especially a casino boss, knows that [the doubling strategy] is a very risky way of winning 1€”. This type of strategy has to be ruled out: there should be a lower bound on the player’s loss” ([4] p. 130). This passage is interesting on several levels. Most obviously, under the received treatment the doubling strategy, as an arbitrage, in fact has no risk—as we will see, under the usual treatment there exists no state in which the agent can lose. In labeling the doubling strategy as risky, Delbaen and Schachermayer make clear, apparently without realizing it, that they reject the implication of the formal characterization of the doubling strategy’s payoff that the doubling strategy is riskless.

In any case, as a general matter there is no need to rule out very risky portfolio strategies. Assuming that agents are risk averse, they will not adopt very risky portfolio strategies even if such strategies are available unless they are well compensated in terms of expected return. Just the opposite: the (apparent) necessity for ruling out the doubling strategy arises precisely from the fact that the doubling strategy is (represented as) not being risky.

As we will see, the formal treatment of the doubling strategy implicit in the extended definition of the stochastic integral involves a specification of the state space that seems questionable. At first glance it appears natural when analyzing the doubling strategy to take the state ω as the date at which the agent first wins ($\omega = 1, 2, \dots$). Doing so, however, has the unfortunate consequence that by definition the event that the agent never wins cannot occur. In other words, the event that gives the doubling strategy its interest is defined out of existence in the formal treatment. Obviously the specification that agents cannot lose under the doubling strategy plays a central role in generating the conclusion that the doubling strategy is an arbitrage.

It is likely that analysts find it acceptable to rule out the event that the agent never wins because this event occurs with probability zero, and zero-probability events are routinely ignored in all applications of probability. Neglecting zero-probability events is acceptable when, as is usual, the payoff that occurs with zero probability is

finite. However, it is not so obvious that infinite (or, as under the doubling strategy, negative infinite) payoffs that occur with zero probability can be ignored without consequence.²

The payoffs of a sequence of finitely repeated doubling strategies terminated at successively later dates involves a negative payoff in the event of not winning that is increasing in absolute value, but occurs with decreasing probability. The former makes the payoff less attractive and the latter makes it more attractive. Under most utility functions that incorporate strict risk aversion and can value arbitrary negative payoffs (quadratic, for example) the horserace between the magnitude of the negative payoff and its probability is won by the latter, despite the former being ahead at all finite dates: investors view the payoff of the doubling strategy viewed as a random variable as worse and worse the longer the doubling strategy is played, as long as only a finite number of plays occur. Thus for all finite time the increasing magnitude of the loss dominates its vanishing probability. However, we are asked to believe that in the limit the situation is reversed and the zero probability dominates, resulting in the doubling strategy becoming an arbitrage.

The alternative characterization of payoffs of infinite portfolios proposed here involves augmenting the set of states to include not just the natural numbers, but also ∞ . In the context of the doubling strategy $\omega = \infty$ represents the event that the agent never wins. As is usual, payoffs of infinite portfolio strategies are modeled as limits of payoffs of finite portfolio strategies. Here, however, the limit is not defined either in terms of the Hilbert norm, as in the Ito calculus, or in a pointwise sense. Rather, the limiting payoff is defined so that its value is the limit of the values of the finite-date payoffs. In the context of the doubling strategy the interpretation is that its (state-contingent) payoff consists of a positive payoff at the date associated with each finite state ω and a negative payoff at ∞ . The net value of these contingent payoffs equals the initial cost of the doubling strategy, which is zero. Because of the negative payoff at ∞ , the doubling strategy is no longer necessarily an arbitrage.

Because the initial cost of a portfolio strategy with nonzero payoff at ∞ differs from the summed values of its finite-date payoffs, such portfolio strategies correspond to rational bubbles. Therefore the revised representation of payoffs of infinite portfolio strategies constitutes a suitable framework for theoretical analysis of models in which portfolio strategies can include bubbles.

²The assumption that justifies the usual treatment corresponds in mathematical terms to the presumption that $0 \times \infty$ can be taken to be equal to 0. Curiously, mathematicians appear not to have adopted a uniform treatment of this question. Wolfram MathWorld and Wikipedia, “Extended Real Number Line” state that $0 \times \infty$ is usually left undefined. The latter adds that in probability and measure theory, in contrast, $0 \times \infty$ is sometimes defined as equaling 0. Aliprantis and Border [1] assumed $0 \times \infty = 0$ along with the other arithmetic rules involving ∞ . Royden [15] listed the assumption $0 \times \infty = 0$ separately from the others, identifying it as an “arbitrary *convention*” (emphasis in original), suggesting that he was not entirely comfortable with the assumption.

The rules of arithmetic for expressions involving ∞ are sometimes justified in terms of limits. None of the above sources presented a justification along these lines for $0 \times \infty = 0$.

2 The Doubling Strategy: Formal Analysis

We begin with a review of the received version of the definition of portfolio payoffs and of the doubling strategy, in which the state ω is taken to be the date at which the gambler first wins ($\omega = 1, 2, 3, \dots$). Since both dates and states are countable, the space of portfolio payoffs and dividends can be taken to be the sets of adapted real-valued functions on $N \times N$, with typical elements $v(t, \omega)$ and $z(t, \omega)$. Here N denotes the natural numbers. A filtration on N describes the resolution of uncertainty. In the context of the doubling strategy the assumed information at t consists of knowledge of the state ω if $\omega \leq t$, and knowledge that $\omega > t$ otherwise. Thus the events at date t can be denoted $\{e_1, e_2, e_3, \dots, e_t, e^t\}$, where $e_\tau = \{\tau\}$ for $\tau = 1, \dots, t$ and $e^t = \{(t+1) \cup (t+2) \cup (t+3) \cup \dots\}$.

Substituting v_t for $v(t, \omega)$ and similarly for z , we have

$$v_t = v_{t-1} + \theta_{t-1}\varepsilon_t - z_t, \quad (5)$$

$t = 1, 2, \dots$. Here θ represents the number of units of a single risky asset purchased and ε represents the return on the risky asset. As (5) implies, any funds not invested in the risky asset or paid out in dividends are assumed to be allocated to a riskless asset, which has a zero net interest rate.

2.1 Finite Portfolio Strategies

A finite portfolio strategy is one for which the number of securities is finite and there exists a date $\tau \in N$ at which the portfolio is terminated by payment of a liquidating dividend z_τ at τ . Equivalently, for a finite portfolio strategy there exists a date τ for which v_t, θ_t and z_t equal zero for $t > \tau$.

A finite version of the doubling strategy is a portfolio strategy for which $v_0 = 0$ and θ_t is given by

$$\theta(t, \omega) = \begin{cases} 2^t, & 0 \leq t < \tau, \quad t < \omega \\ 0, & \text{otherwise} \end{cases}. \quad (6)$$

It is assumed without material loss of generality that ε is distributed as IID Bernoulli, and is a fair game:

$$\varepsilon_t = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases} \quad (7)$$

(for analysis of the doubling strategy when equilibrium prices reflect risk aversion see Fisher and Gilles [7]). The state ω is taken to be the first date at which $\varepsilon_t = 1$,

implying that state ω occurs with probability $2^{-\omega}$. The assumption of risk neutrality implies that the risk-neutral pricing measure coincides with the natural measure of probability. These specifications imply that the dividend of the finite doubling strategy is

$$z(t, \omega) = \begin{cases} 1 & \text{for } t = \omega \leq \tau \\ -2^\tau + 1 & \text{for } t = \tau < \omega \\ 0 & \text{otherwise} \end{cases} . \quad (8)$$

Define the gain $Z_t(\omega)$ as $\sum_{i=1}^t z(i, \omega)$. Then for any $t \leq \tau$ the distribution of Z_t is

$$Z_t = \begin{cases} 1 & \text{with probability } 1 - 2^{-t} \\ -2^t + 1 & \text{with probability } 2^{-t} \end{cases} . \quad (9)$$

implying that $E(Z_t) = 0$. Further, we have $Z_t = E_t(Z_\tau)$, implying that $\{Z_t\}$ is a martingale.

2.2 Infinite Portfolio Strategies

If for any portfolio strategy $\{\theta\}$ the portfolio weights obey the restriction

$$\sum_{t=0}^{\infty} E(\theta_t^2) < \infty. \quad (10)$$

the payoffs associated with θ truncated at date τ converge in mean-square to a finite-variance random variable as τ becomes large. However, the condition is not satisfied by the doubling strategy. For termination date τ the distribution of θ_τ implied by (6) is

$$\theta_\tau = \begin{cases} 2^\tau & \text{with probability } 2^{-\tau} \\ 0 & \text{with probability } 1 - 2^{-\tau} \end{cases} , \quad (11)$$

implying that $E(\theta_\tau^2) = 2^\tau$. Therefore $\sum E(\theta_\tau^2)$ rises without limit.

The same conclusion can be derived by computing the variance of the gain. For termination date τ the distribution of the gain is

$$Z_\tau = \begin{cases} 1 & \text{with probability } 1 - 2^{-\tau} \\ -2^\tau + 1 & \text{with probability } 2^{-\tau} \end{cases} , \quad (12)$$

from which it is easily checked that the variance of Z_τ goes to infinity with τ . In this case the extended definition of the stochastic integral implies that the gain on the

infinitely repeated doubling strategy consists of the pointwise limit of the gains on the truncated doubling strategies, which is 1 with certainty. This is so because under pointwise convergence the portfolio strategy is terminated only upon success, and eventual success is inevitable under the adopted characterization of the state space.

There is no question that this demonstration is formally correct, but it depends on the validity of a particular construction of the payoff of the infinite portfolios. There are no theoretical grounds for preferring pointwise convergence (under which the doubling strategy is an arbitrage) to mean-square convergence (under which the doubling strategy is not a well-defined portfolio strategy) or to any other convergence criterion. Further, the reader is reminded that in the foregoing analysis the event of never winning was ruled out by assumption in the specification of the state space, which comes very close to directly assuming the conclusion that the payoff of the doubling strategy is an arbitrage.

In the remainder of this paper we will present a different construction of the payoffs of infinite portfolio strategies. Under the alternative construction the doubling strategy has a well-defined payoff, in contrast to the case under mean-square convergence. Whether this payoff is an arbitrage depends on agents' preferences. This is an attractive feature of the revised analysis. In view of the definition of an arbitrage (as a portfolio strategy with a payoff strictly preferred to zero and a nonpositive initial cost), one must be suspicious of a purported demonstration of arbitrage that makes no reference to agents' preferences, as has been the case up to now.

3 The Doubling Strategy: An Alternative Formalization ³

We saw in the preceding section that, on the received construction of payoffs of infinite portfolio strategies, arbitrage in the doubling strategy arises because the possibility of never winning is ruled out by assumption. Remedying this misspecification involves defining a setting in which the event of never winning is explicitly represented, and modeling it in such a way that this event affects the payoff even though it occurs with probability zero. This involves modeling payoffs that have components that occur in the infinite future, and requires a setting in which these are limits of finite-date payoffs. In such a setting we will find that sequences of finite payoffs may converge to payoffs that have nonzero components at infinity. Specifically, the payoff of the infinitely repeated doubling strategy involves a negative component at infinity in addition to positive components at the finite dates.

Achieving these goals involves specifying a payoff index set that is compact. The natural numbers are not compact in the topology to be specified. We therefore append ∞ to the natural numbers, with ∞ interpreted as the state in which the agent never wins. In the topology to be specified, ∞ is a limit point of N , so we

³This section draws heavily on Fisher and Gilles [7].

have the Alexandroff one-point compactification. When payoffs are constructed as function spaces based on the compactified natural numbers, they turn out to have the desired properties.

The compactified version of the natural numbers N is denoted N_∞ , where $N_\infty \equiv N \cup \{\infty\}$. We will define payoffs of portfolio strategies as measures on $N_\infty \times N_\infty$. We define a metric d on $N_\infty \times N_\infty$ by

$$d((t, \omega), (t', \omega')) = \sqrt{(2^{-t} - 2^{-t'})^2 + (2^{-\omega} - 2^{-\omega'})^2}, \quad (13)$$

where $0 < t, \omega, t', \omega' \leq \infty$. This metric coincides with Euclidean distance if one associates $1, 2, 3, \dots$ with $1/2, 3/4, 7/8, \dots$ for both t and ω , and associates ∞ with 1 . Under the topology associated with the distance measure just defined $N_\infty \times N_\infty$ is compact.

A basis for the topology associated with the distance measure (13) consists of the pairs (x, y) , where x consists of the elements of N_∞ and y consists of the unions of the singleton subsets of N (not N_∞) and the tails of N_∞ (that is, the unions of ∞ and the elements of N beyond some $n \in N$). Under this topology, loosely, (∞, ∞) is almost indistinguishable from the elements of $N \times N$ that are very large in both components. We will see below that this choice of topology will ensure that infinite portfolio strategies have well-defined payoffs under weak boundedness conditions, and that the payoffs of sequences of finite portfolio strategies converge to these payoffs.

Let C be the space of continuous real-valued functions on $N_\infty \times N_\infty$.⁴ Under the topology on $N_\infty \times N_\infty$ just specified, C consists of the functions $f : N_\infty \times N_\infty \rightarrow \mathcal{R}$ such that $\lim_{\omega \rightarrow \infty} f(t, \omega) = f(t, \infty)$, $\lim_{t \rightarrow \infty} f(t, \omega) = f(\infty, \omega)$, and $\lim_{t, \omega \rightarrow \infty} f(t, \omega) = f(\infty, \infty)$, for $t, \omega \in N$. C is equipped with the topology induced by the norm $\|f\|_\infty = \sup_{t, \omega} |f(t, \omega)|$, $t, \omega \in N_\infty$.

Payoffs of portfolio strategies will be modeled as elements of $M(N_\infty \times N_\infty)$, the space of finite signed measures on the Borel σ -field $B(N_\infty \times N_\infty)$. We define on $M(N_\infty \times N_\infty)$ the topology of weak* convergence, so that, if μ is a net of measures, we have $\mu_n \rightarrow \mu$ if and only if

$$\int_{N_\infty \times N_\infty} f d\mu_n \rightarrow \int_{N_\infty \times N_\infty} f d\mu \quad (14)$$

for all $f \in C$. Then C and M are topological duals under

$$\langle f, \mu \rangle = \int_{N_\infty \times N_\infty} f d\mu \quad (15)$$

for $f \in C$, $\mu \in M$.

⁴For economists, the best source for the mathematics used in this paper is Aliprantis and Border [1].

Payoffs of finite portfolio strategies are represented by the elements of $M(N_\infty \times N_\infty)$ that are zero after some finite date and state. The filtration on N defined above applies to N_∞ with the proviso that e^t is assumed to include ∞ as well as $\{t+1, t+2, \dots, t+i, \dots\}$ for finite i . The σ -algebra at $t = \infty$ is taken to be the power set of N_∞ , so that at ∞ agents know ω .

The payoff of an infinite portfolio strategy will be represented as the weak* limit of the payoffs of a sequence of finite portfolio strategies. Most simply, consider the sequence of payoffs of finite portfolio strategies represented by the Dirac measures $\delta_{i,j}$ defined by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } t, e = i, j \\ 0 & \text{if } t, e \neq i, j \end{cases} \quad (16)$$

for e an event at t (i.e., $e \subset \{1, 2, \dots, t, e^t\}$). Here $\delta_{i,j}$ indicates one unit of output delivered at date i and event j , 0 otherwise. It is seen that the space of payoffs of finite portfolio strategies is the span of the $\delta_{i,j}$ for $i, j \in N$.

As a special case, $\delta_{\infty, \infty}$ denotes one unit of output delivered at date ∞ and state ∞ . Imposition of the weak* topology on $N_\infty \times N_\infty$ implies that as n approaches ∞ , $\{\delta_{n,n}\}$ converges to $\delta_{\infty, \infty}$. To see this, take any test function f from C . Then

$$\langle f, \delta_{n,n} \rangle = f(n, n) \text{ and } \langle f, \delta_{\infty, \infty} \rangle = f(\infty, \infty). \quad (17)$$

Because f is continuous, $\{f(n, n)\}$ converges to $f(\infty, \infty)$, implying that $\langle f, \delta_{n,n} \rangle$ converges to $\langle f, \delta_{\infty, \infty} \rangle$, as required for weak* convergence of $\delta_{n,n}$ to $\delta_{\infty, \infty}$.

Alaoglu's theorem implies that any bounded net of payoffs of finite portfolio strategies contains a convergent subnet. It is assumed here that the limit point is unique. It follows that all infinite portfolio strategies that are of interest, including the doubling strategy, have well-defined payoffs.

3.1 Infinite Portfolio Strategies

In Subsection 2.1 we interpreted portfolio payoffs of finite portfolio strategies as functions on $N \times N$. They can equally well be taken to be functions on $N_\infty \times N_\infty$ (with, of course, zeros in the row and column associated with ∞). To convert to measures we define a particular measure ν on N_∞ as the numeraire, where for each measurable event e at date $t \in N_\infty$, $\nu_{t,e}$ is defined as the probability of e occurring. Then the payoff of any finite portfolio strategy interpreted as a measure can be derived by applying its payoff defined as a function as specified above to the numeraire measure ν . Here the portfolio payoff interpreted as a function is the Radon-Nikodym derivative with respect to ν of the payoff interpreted as a measure. Given that we are assuming risk-neutral valuation, multiplying by probabilities implies that payoffs at particular

events are measured by their date-0 values.⁵

As an example, consider the doubling strategy terminated at date 2. Interpreted as a measure μ_2 its payoff is given by

$$\mu_2 = \frac{1}{2}\delta_{1,1} + \frac{1}{4}\delta_{2,2} - \frac{3}{4}\delta_{2,e^2}. \quad (18)$$

Correspondingly, the payoff of the doubling strategy terminated at date T is

$$\mu_T = \sum_{t=1}^T 2^{-t}\delta_{t,t} - (1 - 2^{-T})\delta_{T,e^T}. \quad (19)$$

The payoff of an infinite portfolio strategy is defined as the weak* limit of the payoffs of the finite portfolio strategies defined by truncating at date T , for an increasing sequence of T . For the doubling strategy, since $(1 - 2^{-T})\delta_{T,e^T}$ weak* converges to $\delta_{\infty,\infty}$ we have

$$\mu_\infty = \sum_{t=1}^{\infty} 2^{-t}\delta_{t,t} - \delta_{\infty,\infty}. \quad (20)$$

Thus the doubling strategy has a (probability-adjusted) payoff of 2^{-t} at t in event t for finite t , and a payoff of -1 at ∞ for $\omega = \infty$. Note that the coefficients sum to zero, the initial cost of the doubling strategy. It is seen that the possibility of losing forever is a component of the payoff of the infinitely repeated doubling strategy even though this event occurs with probability zero.

Observe that for the infinitely repeated doubling strategy μ_∞ cannot be constructed by applying a Radon-Nikodym derivative to ν : since ν takes value 0 at (∞, ∞) while $\mu_\infty \neq 0$ at (∞, ∞) , μ_∞ is discontinuous with respect to ν .

We defined infinite portfolio strategies as portfolio strategies under which an infinite number of nonzero transactions occur. Some infinite portfolio strategies have the property that the number of securities traded at each of an infinite number of events is finite. The doubling strategy and most of the infinite portfolio strategies considered in this paper are in this class. For such portfolio strategies there is no ambiguity about initial cost: the initial cost equals the sum over securities of the position in each security at the initial date. However, other infinite portfolio strategies involve trading an infinite number of securities at the initial date. It is natural to assume that the initial cost of such portfolio strategies equals the infinite sum of the values of the initial position. However, as Werner [17] has shown, payoff pricing functionals with this property can involve arbitrage. In Subsection 4.2 we will provide an

⁵We could have adopted the same numeraire choice in defining portfolio payoffs in the preceding section. Instead we defined payoffs there in terms of contemporaneous value. That choice had the convenient implication that the payoff of the finitely repeated doubling strategy consists of two values (when nonzero). Here, in contrast, the doubling strategy terminated at date T takes on $T+1$ values. The present specification has the attractive property that the payoff of the doubling strategy when unsuccessful converges to -1 rather than $-\infty$.

example—essentially the same as that in Werner—in which the initial cost of infinite portfolio strategies strictly exceeds the indicated infinite sum.

3.2 Arbitrage under Infinite Portfolio Strategies

An arbitrage is a portfolio strategy with a nonpositive initial cost and a payoff z that an agent strictly prefers to the zero payoff (here, following Kreps [12], we are taking the origin as the agent’s endowment point), or a strictly negative initial cost and a positive payoff. For finite portfolio strategies a sufficient condition on preferences that implies that z is strictly preferred to zero is usually that z is strictly positive (i.e., positive and nonzero). As we have seen, for any termination date the payoffs of the finitely repeated doubling strategy sum to zero, implying that there is no arbitrage.

The fact that the payoffs of infinite portfolio strategies are constructed here as weak* limits implies that the existence or nonexistence of arbitrage depends on whether preferences are weak* continuous. If they are, then the fact that the set of payoffs strictly preferred to zero is open implies that if there exists arbitrage on payoffs of infinite portfolio strategies then there exists arbitrage on payoffs of nearby finite portfolio strategies. It therefore follows from the fact that there is no arbitrage in finite portfolio strategies that the same is true for infinite portfolio strategies. Thus neither the finitely repeated nor the infinitely repeated doubling strategy produces an arbitrage.

If $z = \sum_{t,e=1}^n z_{t,e} \delta_{t,e}$ for finite n , as is the case with finite portfolio strategies, a utility function consistent with risk-neutral valuation of payoffs of finite portfolio strategies is

$$U(z) = \sum_{t,e=1}^n z_{t,e} \tag{21}$$

(recall the probabilities have already been incorporated in the definition of payoffs as measures). This utility function implies that agents are indifferent between implementing the finitely repeated doubling strategy and not doing so. A continuous extension of U to a function with domain $N_\infty \times N_\infty$ is given by

$$U(z) = \sum_{t,e=1}^{\infty} z_{t,e} + \sum_{t=1}^{\infty} z_{t,\infty} + \sum_{e=1}^{\infty} z_{\infty,e} + z_{\infty,\infty}. \tag{22}$$

Under this specification agents are indifferent between implementing infinite portfolio strategies, finite portfolio strategies and the zero portfolio strategy, as expected under risk neutrality and fair pricing of securities.

In general, if agents are strictly risk averse over finite portfolio strategies, an extension of U to a function with domain $N_\infty \times N_\infty$ that by construction preserves continuity, and therefore continuous valuation, is achieved by defining the utility of the payoff of any infinite portfolio as the limit of the utilities of payoffs of a

sequence of finite portfolio payoffs that converges to that payoff. The utility function (22) has this property. By that definition, the infinitely repeated doubling strategy has negative infinite utility under many common utility functions that incorporate strict risk aversion, such as quadratic. In this case the formal representation of the doubling strategy corresponds to Delbaen and Schachermayer’s observation that the doubling strategy is “a very risky way of winning 1€”, discussed in the introduction. If agents are strictly risk averse but have sufficiently low risk aversion—are “nearly risk neutral”, in the terminology of Fisher and Gilles [7]—then $U(x)$ is a negative number, but a finite one. If agents are strictly risk averse they strictly prefer the zero portfolio strategy to either the finitely repeated doubling strategy or the infinitely repeated doubling strategy, and this is so whether or not they are nearly risk neutral.

In contrast, one can define the utility of the payoff of an infinite portfolio strategy as the utility of its finite-date and finite-state components. Under this assumption the payoff at ∞ does not affect utility. This involves defining the set of payoffs strictly preferred to zero as the set of finite-date and finite-state payoffs strictly greater than zero combined with any payoff at infinity). As the doubling strategy indicates, this extension implies weak* discontinuous valuation (unless, of course, portfolio strategies like the doubling strategy are subject to ad hoc restrictions such as lower bounds). Because the component of portfolio payoffs at infinity does not affect utility, there exists the possibility of arbitrage in infinite portfolio strategies under this extension of utility even when finite portfolio strategies are free of arbitrage. This modeling procedure produces the same analysis of the doubling strategy as that necessarily produced under the received analysis by the exclusion of payoffs at infinity. Thus the analyst using the representation of infinite portfolio strategies proposed here has the option of accepting the received account. Here, however, to obtain the conventional account the analyst has to explicitly assume that agents believe that they can entirely avoid a debt by postponing its repayment indefinitely. This seems like a farfetched assumption—and the fact that it generates discontinuous utility and discontinuous valuation is unattractive—but there is no denying that it is an available specification.

We take the view that using weak* convergence to construct payoffs of infinite portfolio strategies and assuming that utility, and therefore valuation, is weak* continuous is a sensible way of modeling portfolio strategies like the doubling strategy. The operative assumption is that the infinite future is qualitatively similar to the very—but finitely—distant future; no implausible discontinuities occur in going from the very distant future to the infinitely distant future. If one believes that investors do not think in a mathematically sophisticated way about the infinite nature of the future, imposition of continuity is the natural choice.

In support of this view, there is no reason to suppose that real-world investors or gamblers believe that they could construct arbitrage strategies in gambling with each other if only they could gamble an infinite number of times, as presumed in the received story. A more reasonable view is that agents do not distinguish between gambling a large finite number of times and gambling an infinite number of times.

If so, the modeling strategy proposed here provides a suitable basis for modeling the behavior of agents who are assumed able to gamble an infinite number of times.

4 Bubbles

The *fundamental value* of the payoff of a portfolio strategy is the summed value of its finite-date components. The *bubble value* is the difference between its initial cost and the fundamental value. If attention is restricted to portfolio strategies without bubbles there is no material difference between portfolio payoffs as conventionally modeled and portfolio payoffs as modeled here: in the latter case the values at the termination date of finite portfolio strategies converge to zero, so that the payoff of infinite portfolio strategies has a zero component at infinity.

Modeling bubble payoffs as weak* limits of the terminal payoffs of finite-date portfolio strategies, as proposed here, is not a new idea (see Gilles [8], Gilles and LeRoy [9], [10], Fisher and Gilles [7], Werner [17]). With the exception of Fisher and Gilles [7], those papers did not compactify the payoff index set. As a result, in those papers the limiting payoff took the form of a finitely additive measure. That is a major disadvantage: these are nonconstructive and, as a result, extremely abstract and difficult to work with. Because they depend for their existence on the Axiom of Choice, they are controversial, at least to some mathematicians. Compactifying the payoff index set and representing payoffs by measures, in contrast, makes possible a representation of the limiting payoff that is just as simple as the representation of finite-date payoffs.

Bubbles come up frequently in financial economics, although the analyst often does not recognize them as such or make the connection with the general theory of bubbles. The analysis proposed here can be used to deal with such cases. Given the similarity between these applications and the material presented above, the models are not presented formally.

4.1 Ponzi Schemes

In a Ponzi scheme an investor borrows money and rolls over the indebtedness forever, implying that, under the received modeling convention, the Ponzi scheme is an arbitrage. The modeling procedure outlined here can be used to model the Ponzi scheme, with the attractive simplification that $N_\infty \times N_\infty$ can be replaced by N_∞ since the problem is deterministic. The payoff of the Ponzi scheme played n times is $-\delta_n$, the Dirac measure introduced above. As n increases $\{-\delta_n\}$ converges to $-\delta_\infty$, implying that the borrower has to pay back the loan at ∞ . Whether or not the Ponzi scheme is an arbitrage depends on preferences, exactly as with the doubling strategy. If agents' utilities are assumed not to depend on the component of consumption at ∞ , then utility is weak* discontinuous and the Ponzi scheme is an arbitrage. In economic terms, the presumption is that agents regard a loan that does not have a specific

repayment date as effectively free money. As with the doubling strategy, this is a substantive economic assumption about how agents view an infinite future, one that may or may not be descriptively accurate.

If, on the other hand, one imposes continuous utility functions, the Ponzi scheme is not an arbitrage: agents view repaying the loan at ∞ as being similar to paying it back at a distant future date. The distinctive feature of the modeling approach propose here is that it enables—or, if one prefers, requires—the analyst to state explicitly how preferences are assumed to extend to the infinite case.

It is seen that Ponzi schemes are essentially similar to the doubling strategy. If agents can operate the infinitely repeated doubling strategy and it is an arbitrage, the same is true of Ponzi schemes. It is difficult to see why agents would bother with the doubling strategy if they could operate Ponzi schemes.

4.2 Equilibrium with Valued Tail Consumption

Consider a deterministic representative-agent setting in which the representative agent can trade securities $x_0, x_1, x_2, x_3, \dots$ defined by payoffs (measured in terms of contemporaneous consumption) as follows:

$$x_0 = (1, 1, 1, \dots) \tag{23}$$

$$x_1 = (1, 0, 0, \dots) \tag{24}$$

$$x_2 = (0, 1, 0, \dots) \tag{25}$$

$$x_3 = (0, 0, 1, \dots) \tag{26}$$

...

(this is the same set of payoffs assumed in Werner [17]). We will refer to x_0 as the bond, as is appropriate if the index set is interpreted as representing time. (In Werner [17] the index set is interpreted as representing states of nature rather than dates, so that x_0 is the risk-free payoff; either interpretation is acceptable for the present purpose.)

The representative agent maximizes the utility of the consumption stream (c_1, c_2, c_3, \dots) , where utility is defined by

$$U = \sum_{t=1}^{\infty} 2^{-t} z_t + \lim_{t \rightarrow \infty} z_t. \tag{27}$$

A finite portfolio strategy is defined by weights $\theta_0, \theta_1, \dots, \theta_T$ for some T and, from (23) -(26), the z_t are related to the θ_t by

$$z_t = \theta_0 + \theta_t, \tag{28}$$

so that (27) can be written as

$$U = 2\theta_0 + \sum_{t=1}^T 2^{-t}\theta_t. \quad (29)$$

Infinite portfolio strategies are the same except that the number of nonzero values of θ_t is not assumed to be finite. In that case, from (27), utility is given by

$$U = 2\theta_0 + \sum_{t=1}^{\infty} 2^{-t}\theta_t + \lim_{t \rightarrow \infty} \theta_t. \quad (30)$$

It is assumed that the θ_t are such that the infinite sum and the limit exist, so that agents can assign an unambiguous utility level to any payoff (see Werner [17] and the papers cited there for discussion of the more general case in which θ can be any bounded sequence).

The utility function (30) also serves as the payoff pricing functional (the functional that assigns portfolio prices to sets of portfolio weights). Here the price of security 0 is 2 and the price of security t , $t = 1, 2, \dots$, is 2^{-t} , equal to the applicable discount factor. As indicated in Subsection 3.1, the initial cost of infinite portfolio strategies does not necessarily equal the infinite sum of the values of the securities, as would be the case in the absence of the limit term.

Consider the portfolio strategy $(\theta_0, \theta_1, \theta_2, \dots) = (1, 0, 0, \dots)$. Its payoff (measured in units of date-0 value, as indicated above) is $\delta_1/2 + \delta_2/4 + \delta_3/8 + \dots + \delta_\infty$. This payoff is the weak* limit as T goes to ∞ of the payoffs of the portfolio strategies consisting of buying the bond at date 0 and selling it at date T . Its initial cost is 2, which equals the infinite sum of the payoffs at t including that at ∞ , as indicated above. The infinite portfolio strategy $(0, 1, 1, 1, \dots)$ consisting of one unit of x_1, x_2, x_3, \dots has the same payoff and the same initial cost as the finite portfolio strategy $(1, 0, 0, \dots)$. If the limit term did not appear in (30) the initial cost of the latter portfolio strategy would be 1 instead of 2, implying that the infinite portfolio strategy $(-1, 1, 1, \dots)$ would have initial cost -1 and payoff $(0, 0, \dots)$, and would therefore be an arbitrage.

Since the payoff of the buy-and-hold portfolio strategy involving the bond has a nonzero payoff at ∞ , the bond has a bubble. Finite portfolio strategies involving the other securities do not have bubbles. However, as the preceding analysis indicates, infinite portfolio strategies involving securities that do not have bubbles may have a bubble.

5 Conclusion

The simplest—and least controversial—contribution of this paper is to raise questions about the easy characterization of the doubling strategy as an arbitrage that has to be avoided by imposing trading restrictions. On the contrary, that conclusion emerges

as a consequence of assumptions and modeling conventions the validity of which is far from obvious. From an analytical point of view this conclusion is good news. If agents are strictly risk averse (and have weak*-continuous preferences) our analysis implies that they will avoid the doubling strategy even if it is available. Therefore there is no need to impose portfolio restrictions to exclude these strategies. Analytically this conclusion is convenient: without trading restrictions choice sets are linear spaces and valuation is linear; with trading restrictions matters are more complicated, even in settings that would otherwise be easy to analyze (see, for example, LeRoy and Werner [14], Ch. 4 and 7).

At a somewhat more controversial level, the analysis here gives a compact—in both senses of the word—analytical vehicle for bubbles. Many analysts point out that there exist theoretical conditions having to do with failure of transversality conditions (see, for example, Santos and Woodford [16] and Huang and Werner [11]) that exclude bubbles. These conditions are sometimes represented as reasonable restrictions. Blanchard and Fischer [2] and others (such as me [13]) have expressed the opposite viewpoint. There exists some empirical evidence to the effect that bubbles appear not to be occurring. As usual, this evidence is not conclusive, particularly in view of the financial events of the past several years, and in any case it only tests for the existence of the simplest sort of bubbles.

We take the view that any strong conclusion in this area is premature. Many phenomena occur in financial markets that are difficult to reconcile with the simplest rational-agent model: one thinks of asset price volatility, the equity premium puzzle, the periodic occurrence of liquidity crises. These events do not appear to produce obvious profit opportunities that agents are irrationally ignoring. Accordingly, it should be possible to analyze them using the orthodox methods of financial economics. If some or all of these phenomena turn out to be connected to bubbles, the analytical techniques proposed here may be useful.

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