Infinite Portfolio Strategies∗

Stephen F. LeRoy
University of California, Santa Barbara
November 15, 2010

Abstract

Continuous-time stochastic calculus extends the definition of the stochastic integral to the case where the integrand is not square-integrable. The extension uses a pointwise limit. Under the extension absence of arbitrage in finite portfolio strategies is consistent with existence of arbitrage in infinite portfolio strategies. The doubling strategy and Ponzi schemes are the most common examples. We argue that this extension may or may not make economic sense, and propose an alternative extension of the definition of the stochastic integral under which absence of arbitrage can be preserved. The extension involves appending a date and state called ∞ to the payoff index set, and defining a topology such that the set of payoffs is compact. Doing so allows a simplified mathematical treatment of a number of topics that are unwieldy when modeled in a setting where the payoff index set is noncompact.

In continuous-time finance theory, portfolio payoffs are defined using the Ito definition of the stochastic integral. Security prices $x$ are modeled as a martingale—most simply, as a Brownian motion—and a portfolio strategy is represented by a predictable stochastic process $\theta$. Assuming that portfolio strategies obey

$$\int_{t=0}^{T} E(\theta_t^2) dt < \infty$$

the definition of the integral implies that for any $\tau \leq T$ the value at time $\tau$ of the portfolio, given by

$$\int_{t=0}^{\tau} \theta_t dx_t,$$
is a random variable with finite mean and variance.

In many expositions it is pointed out that the definition of the Ito stochastic integral, and with it the characterization of payoffs of portfolio strategies, can be extended to cases in which the condition (1) is not satisfied. To see this, suppose that (1) fails, but the weaker condition

$$\int_{t=0}^{\tau} E(\theta_t^2) dt < \infty$$  

(3)

is satisfied for all $\tau < T$. In that case the integral

$$\int_{t=0}^{\tau} \theta_t dx_t$$  

(4)

can be defined as a random variable for each $\tau < T$, and therefore for each state $\omega$ the integral (2) can be defined as the limit of (4) at state $\omega$ when $\tau$ approaches $T$ (a similar extension of the definition is available when $x$ is assumed to be a local martingale rather than a martingale; we will not discuss this case).

Neither the mathematics-oriented sources for this extension (for example, Chung and Williams [3]) nor the finance-oriented sources (for example, Duffie [5], Appendix D) provide much in the way of motivation for identifying the extension with a pointwise limit. The reader is left uncertain whether the extension of the integral involves economic assumptions beyond those involved in the standard case. We look into this question in the present paper, taking the doubling strategy as an example of a portfolio strategy for which the extended definition of the stochastic integral must be employed. We find that, at least in the context of the doubling strategy, the extension is indeed questionable in a sense that the original integral (2) is not.

The extension just defined turns out to imply a possible discontinuity in the valuation of portfolio gains and, under weak assumptions, the utilities that these gains generate: the values of gains of portfolio strategies for $\tau < T$ do not converge to those of portfolio strategies at $T$ as $\tau$ rises toward $T$. We take the view that there is no economic motivation for this discontinuity, and its existence suggests a misspecification.

As just noted, the classic example of the discontinuity in valuation just described is the doubling strategy. Truncated versions of the doubling strategy are increasingly risky, and therefore are increasingly unattractive to risk-averse agents. However, under the usual treatment the infinite version of the doubling strategy is an arbitrage. We question the appropriateness of this discontinuity in valuation in the context of the doubling strategy. If our argument for the unsuitability of the received treatment of the doubling strategy is accepted, there are (at least) three possible ways to proceed. The first is to revoke the status as bona fide portfolio strategies of candidate portfolio strategies that do not satisfy the condition (1), even when they do satisfy the
weaker restriction (3). Thus we simply assume that investors do not choose candidate portfolio strategies for which (1) is not satisfied. Arguably this is more an evasion of the problem than a solution to it. The second way to proceed is to adopt a definition of the state space that recognizes the event of never winning as a possible outcome. It turns out that, even though never winning is a zero-probability event, its existence affects the characterization of payoffs of infinite portfolio strategies. It turns out that the infinitely repeated doubling strategy may or may not be an arbitrage depending on how utility functions are specified.

The third procedure involves a more fundamental reformulation of the definition of convergence. Instead of defining the portfolio gain on infinite portfolio strategies as the pointwise limit of the gains of the associated finite portfolio strategies, we take limits on the portfolio dividends themselves. In other words, we take the limit of sequences of stochastic processes rather than sequences of random variables. Doing so makes it possible to interpret dividends at finite but distant dates as converging to dividends “at infinity”. The second and third modifications lead to the same substantive analysis of the doubling strategy: unlike under the received treatment, the doubling strategy may or may not be an arbitrage depending on what one assumes about utility functions. However, we will see that working with convergence of stochastic processes—dividends—instead of convergence of random variables—gains—makes it possible to make the connection with bubbles and Ponzi schemes.

1 The Doubling Strategy

As noted, the prototype candidate portfolio strategy for which the condition (1) fails is the doubling strategy, under which the investor doubles the investment in a risky asset until he generates a gain, and then stops. The doubling strategy is discussed in practically every introductory graduate-level text in financial economics, but most discussions fail to make the connection with the stochastic integral (for example, Etheridge [6]).

Because the portfolio weights under the doubling strategy do not satisfy (1), in defining the payoff of the infinitely repeated portfolio strategy it is necessary to employ the extension of the definition of the stochastic integral described above: the payoff of the infinitely repeated doubling strategy is defined as the pointwise limit of the payoffs of the truncated doubling strategies. This limit envisions the investor gaining $1 with probability 1, an arbitrage. In other words, a portfolio strategy that is increasingly unattractive to risk-averse investors when terminated in finite time morphs into a riskless arbitrage in infinite time. Such arbitrages, it is held, must be ruled out by restrictions on admissible portfolio strategies.

This specification does not correspond in any obvious way to how actual investors behave. Therefore there appears to be no reason to introduce such discontinuities in valuation in analyses that are intended to have empirical relevance. Some analysts, to their credit, appear to be uneasy about labeling the doubling strategy an arbitrage,
particularly in the absence of any characterization of preferences. For example, Delbaen and Schachermayer [4] concluded from the fact that losses are unbounded under the doubling strategy that “[e]verybody, especially a casino boss, knows that [the doubling strategy] is a very risky way of winning 1€. This type of strategy has to be ruled out: there should be a lower bound on the player’s loss” ([4] p. 130). This passage is interesting on several levels. Most obviously, under the received treatment the doubling strategy, as an arbitrage, in fact has no risk—as we will see, under the usual treatment there exists no state in which the agent can lose. In labeling the doubling strategy as risky, Delbaen and Schachermayer make clear, apparently without realizing it, that they reject the implication of the formal characterization of the doubling strategy’s payoff that the doubling strategy is riskless.

In any case, as a general matter there is no need to rule out very risky portfolio strategies. Assuming that agents are risk averse, they will not adopt very risky portfolio strategies even if such strategies are available unless they are well compensated in terms of expected return. Just the opposite: the (apparent) necessity for ruling out the doubling strategy arises precisely from the fact that the doubling strategy is (represented as) not being risky.

As we will see, the formal treatment of the doubling strategy implicit in the extended definition of the stochastic integral involves a specification of the state space that seems questionable. At first glance it appears natural when analyzing the doubling strategy to take the state $\omega$ as the date at which the agent first wins ($\omega = 1, 2, ...$), since then the state probabilities are $1/2, 1/4, ..., 1$. Doing so, however, has the unfortunate consequence that by definition the event that the agent never wins cannot occur. In other words, the event that gives the doubling strategy its interest is defined out of existence in the formal treatment. Obviously the specification that agents cannot lose under the doubling strategy plays a central role in generating the conclusion that the doubling strategy is an arbitrage.

It is likely that analysts find it acceptable to rule out the event that the agent never wins because this event occurs with probability zero, and zero-probability events are routinely ignored in all applications of probability. Neglecting zero-probability events is acceptable when, as is usual, the payoff that occurs with zero probability is finite. However, it is not so obvious that infinite (or, as under the doubling strategy, negative infinite) payoffs that occur with zero probability can be ignored without consequence.\footnote{The assumption that justifies the usual treatment corresponds in mathematical terms to the presumption that $0 \times \infty$ can be taken to be equal to 0. Curiously, mathematicians appear not to have adopted a uniform treatment of this question. Wolfram MathWorld and Wikipedia, “Extended Real Number Line” state that $0 \times \infty$ is usually left undefined. The latter adds that in probability and measure theory, in contrast, $0 \times \infty$ is sometimes defined as equaling 0. Aliprantis and Border [1] assumed $0 \times \infty = 0$ along with the other arithmetic rules involving $\infty$. Royden [14] listed the assumption $0 \times \infty = 0$ separately from the others, identifying it as an “arbitrary convention” (emphasis in original), suggesting that he was not entirely comfortable with the assumption. The rules of arithmetic for expressions involving $\infty$ are sometimes justified in terms of limits.}
The payoffs of a sequence of finitely repeated doubling strategies terminated at successively later dates involves a negative payoff in the event of not winning that is increasing in absolute value, but occurs with decreasing probability. The former makes the payoff, viewed as a random variable, less attractive and the latter makes it more attractive. Assuming a utility function that incorporates strict risk aversion and can value arbitrary negative payoffs (quadratic or negative exponential), under the usual treatment the former dominates in finite time, in the sense that investors regard the payoff viewed as a random variable increasingly unattractive. However, in infinite time the fact that the probability of loss converges to zero dominates. Put differently, for all finite time the increasing magnitude of the loss dominates its vanishing probability. However, in the limit the situation is reversed and the zero limiting probability dominates, resulting in the doubling strategy becoming an arbitrage.

1.1 Finite Portfolio Strategies

We begin with a review of the received version of the definition of portfolio payoffs and of the doubling strategy, in which the state $\omega$ is taken to be the date at which the gambler first wins ($\omega = 1, 2, 3, \ldots$). Since both dates and states are countable, the space of portfolio payoffs and dividends can be taken to be the sets of adapted real-valued functions on $N \times N$, with typical elements $v(t, \omega)$ and $z(t, \omega)$. Here $N$ denotes the natural numbers. A filtration on $N \times N$ describes the resolution of uncertainty. In the context of the doubling strategy the assumed information at $t$ consists of knowledge of the state $\omega$ if $\omega \leq t$, and knowledge that $\omega > t$ otherwise. Thus the events at date $t$ can be denoted $\{1, 2, \ldots, t, e^t\}$, where $e^t = \{(t+1) \cup (t+2) \cup (t+3) \cup \ldots\}$.

Substituting $v_t$ for $v(t, \omega)$ and similarly for $z_t$, we have

$$v_t = v_{t-1} + \theta_{t-1} \varepsilon_t - z_t,$$

(5)

$t = 1, 2, \ldots$. Here $\theta$ represents the number of units of a single risky asset purchased and $\varepsilon$ represents the return on the risky asset. As (5) implies, any funds not invested in the risky asset or paid out in dividends are assumed to be allocated to a riskless asset, which has a zero net interest rate.

A finite portfolio strategy is one for which the number of securities is finite and there exists a date $\tau \in N$ at which the portfolio is terminated by payment of a liquidating dividend $z_\tau$ at $\tau$. Equivalently, for a finite portfolio strategy there exists a date $\tau$ for which $v_t$, $\theta_t$, and $z_t$ equal zero for $t > \tau$.

A finite version of the doubling strategy is a portfolio strategy for which $v_0 = 0$ and $\theta_t$ is given by

None of the above sources presented a justification along these lines for $0 \times \infty = 0$. 

5
Here $\tau$ is the termination date. It is assumed without material loss of generality that $\varepsilon$ is distributed as IID Bernoulli, and is a fair game:

$$
\varepsilon_t = \begin{cases} 
1 & \text{with probability } 1/2 \\
-1 & \text{with probability } 1/2
\end{cases}
$$

(7)

(for analysis of the doubling strategy when equilibrium prices reflect risk aversion see Fisher and Gilles [7]). The state $\omega$ is taken to be the first date at which $\varepsilon_t = 1$, implying that state $\omega$ occurs with probability $2^{-\omega}$. The assumption of risk neutrality implies that the risk-neutral pricing measure coincides with the natural measure of probability. These specifications imply that the dividend of the finite doubling strategy is

$$
\theta(t, \omega) = \begin{cases} 
0, & \omega \leq t \text{ and } t \leq \tau \\
2^t, & \omega > t, t \leq \tau \\
0, & t > \tau.
\end{cases}
$$

(6)

Define the gain $Z_t(\omega)$ as $\sum_{i=1}^{t} z(i, \omega)$. Then for any $t \leq \tau$ the distribution of $Z_t$ is

$$
Z_t = \begin{cases} 
1 & \text{with probability } 1 - 2^{-t} \\
-2^t + 1 & \text{with probability } 2^{-t}
\end{cases}
$$

(8)

implying that $E(Z_t) = 0$. Further, we have $Z_t = E_t(Z_\tau)$, implying that $\{Z_t\}$ is a martingale.

### 1.2 Infinite Portfolio Strategies

If for any portfolio strategy $\theta$ the portfolio weights obey the restriction

$$
\sum_{t=0}^{\infty} E(\theta_t^2) < \infty.
$$

(10)

the gains associated with $\theta$ truncated at date $\tau$ converge in mean-square to a finite-variance random variable as $\tau$ becomes large. However, the condition is not satisfied...
by the doubling strategy. For termination date $\tau$ the distribution of $\theta_{\tau}$ implied by (6) is

$$\theta_{\tau} = \begin{cases} 2^\tau & \text{with probability } 2^{-\tau} \\ 0 & \text{with probability } 1 - 2^{-\tau} \end{cases},$$

(11)

implying that $E(\theta_{\tau}^2) = 2^\tau$. Therefore $\sum E(\theta_i^2)$ rises without limit.

The same conclusion can be derived by computing the variance of the gain. For termination date $\tau$ the distribution of the gain is

$$Z_{\tau} = \begin{cases} 1 & \text{with probability } 1 - 2^{-\tau} \\ -2^\tau + 1 & \text{with probability } 2^{-\tau} \end{cases},$$

(12)

from which it is easily checked that the variance of $Z_{\tau}$ goes to infinity with $\tau$. In this case the extended definition of the stochastic integral implies that the gain on the infinitely repeated doubling strategy consists of the pointwise limit of the gains on the truncated doubling strategies, which is 1 with certainty. This is so because under pointwise convergence the portfolio strategy is terminated only upon success, and eventual success is inevitable under the adopted characterization of the state space.

#### 1.3 Extending the State Space

The simplest way to reverse the conclusion that valuation is discontinuous under the doubling strategy is to add $\infty$ as a date and a state. The date $\infty$ can be interpreted as a dividend that is paid in the infinite future, while the state $\infty$ represents the event of never winning. Thus stochastic processes are real-valued functions on $N_\infty \times N_\infty$, where $N_\infty = N \cup \{\infty\}$. Agents are assumed to know $\omega$ at date $\infty$. In order to avoid payoff sequences that diverge, we will renormalize gains of portfolio strategies by multiplying them by the associated probabilities. Denoting the renormalized gain by $\hat{Z}_{\tau}$, we have, from (12), that the gain on the doubling strategy truncated at $\tau$ is

$$\hat{Z}_{\tau} = \begin{cases} 2^{-\omega}, & \omega \leq \tau \\ -2^\tau + 1, & \omega > \tau \end{cases},$$

(13)

so that the sum of the gains on $\hat{Z}_{\tau}$ over the events at $\tau$ is zero (for example, with $\tau = 2$ the gain is $1/2$ at $\omega = 1$, $1/4$ at $\omega = 2$ and $3/4$ at $\omega^2 = 3 \cup 4 \cup 5 \cup \ldots \cup \infty$.

In this setting the gain on infinite portfolio strategies for state $\omega$ is defined as the limiting gain on the truncated portfolio strategies for state $\omega$. For the doubling strategy we have

$$\hat{Z}_\infty = \begin{cases} 2^{-\omega}, & \omega < \infty \\ -1, & \omega = \infty \end{cases}.$$
As with the truncated doubling strategy, we have that the gains sum to zero. The new element here, of course, is the payoff at \( t = \omega = \infty \), which does not disappear despite the fact that the relevant probabilities converge to zero.

So far we have done nothing more than change the definition of gains on infinite portfolio strategies. Determining what difference, if any, this change makes in the substantive analysis of equilibrium in financial markets depends on how utility functions are extended from \( N \times N \) to \( N_\infty \times N_\infty \). One possible extension involves ignoring the dividends at \( t = \infty \) and \( \omega = \infty \), so that individuals value only dividends that are paid at finite dates and at events with strictly positive probability. This specification amounts to accepting the received characterization of the doubling strategy as an arbitrage, implying the necessity of imposing trading restrictions to rule out the doubling strategy. Even here, however, the revised treatment of the doubling strategy results in a change in interpretation: rather than viewing the doubling strategy as being an arbitrage by its very nature, the revised view is that it is an arbitrage by virtue of a particular characterization of utility functions. In other words, the analyst has a choice about it, and existence of this choice entails the necessity of defending the adopted specification.

Another option is to define the utility generated by infinite portfolio strategies as the limit of the utilities generated by the truncated portfolios the gains of which converge to that of the infinite portfolio strategy. By definition this extension preserves continuous valuation. Under quadratic utility the limiting utility of the doubling strategy is \(-\infty\). In this case there is no necessity for analyst to rule out the doubling strategy: the agents being modeled reject it as prohibitively risky. Under this extension, under any utility function that implies strict risk aversion agents will reject the doubling strategy in favor of investment in the risk-free asset (because the version of the doubling strategy set out here presumes fair pricing).

## 2 Infinite Portfolio Strategies via Convergence of Processes

### 2.1 The Deterministic Case

We now take the analysis in a different direction. There is no inconsistency between the analysis to be presented and that offered above, but the discussion here is useful because it presents similar ideas in a different light. Instead of defining infinite portfolio strategies via convergence of gains, we will characterize dividend streams of infinite portfolio strategies by taking limits of the dividend streams of finite portfolio strategies. In other words, as noted in the introduction, the focus is shifted from sequences of random variables to sequences of stochastic processes. In this section we adopt a deterministic setting, which makes it possible to set out the major ideas in the simplest possible environment. We choose a definition of convergence that
implies that sequences of dividends at finite dates converge to dividends at infinity, exactly the treatment suggested by the preceding discussion. As an example we then consider Ponzi schemes.

Specifying a deterministic setting allows us to define dividends as measures on \( N_\infty \), with the variable interpreted as time. These are assumed to be measurable with respect to the filtration defined above. Dividends on finite portfolio strategies are simple measures (that is, finite sums of Dirac delta functions). We define a norm on \( N_\infty \) by

\[
d(t,t') = \sqrt{(2^{-t} - 2^{-t'})^2}, \quad t,t' \in N_\infty,
\]

which corresponds to Euclidean distance when 1, 2, 3, ..., \( \infty \) are mapped onto 1/2, 3/4, 7/8, ..., 1. In this topology, \( \infty \) is a limit point of \( N_\infty \), and it is the only limit point, so we have the Alexandroff one-point compactification.

Let \( C \) be the space of continuous real-valued functions on \( N_\infty \). Under the topology on \( N_\infty \) generated by the metric (15), \( C \) consists of the functions \( f : N_\infty \to \mathbb{R} \) such that \( \lim_{t \to \infty} f(t) = f(\infty) \). The point of this specification is to ensure that high values of \( t \) are close to \( \infty \), which will enable us to identify measures at the date \( \infty \) as elements of the dividend processes on infinite portfolio strategies.

Dividends of portfolio strategies will be modeled as elements of \( \mathcal{M}(N_\infty) \), the space of finite signed measures on \( N_\infty \). We define on \( \mathcal{M}(N_\infty) \) the topology of weak convergence, so that if \( \mu \) is a net of measures we have \( \mu_n \to \mu \) if and only if

\[
\int_{N_\infty} f d\mu_n \to \int_{N_\infty} f d\mu
\]

for all \( f \in C \). Then \( C \) and \( \mathcal{M} \) are topological duals under

\[
\langle f, \mu \rangle = \int_{N_\infty} f d\mu
\]

for \( f \in C, \mu \in \mathcal{M} \).

The dividend of an infinite portfolio strategy will be represented as the weak limit of the dividend of the associated sequence of finite portfolio strategies. Most simply, consider the sequence of payoffs of finite portfolio strategies represented by the Dirac measures \( \delta_t \) defined by

\[
\delta_t(A) = \begin{cases} 
1 & \text{if } t \in A \\
0 & \text{if } t \notin A.
\end{cases}
\]

\[\text{For economists, the best source for the mathematics used in this paper is Aliprantis and Border [1].}\]
Here $\delta_t$ indicates one unit of output delivered at date $t$, and 0 at all other dates. It is seen that the space of payoffs of finite portfolio strategies is the span of the $\delta_t$. As a special case, $\delta_\infty$ denotes one unit of output delivered at date $\infty$. Imposition of the weak topology on $N_\infty$ implies that as $t$ approaches $\infty$, $\{\delta_t\}$ converges to $\delta_\infty$. To see this, take any test function $f$ from $C$. Then

$$<f, \delta_t> = f(t) \quad \text{and} \quad <f, \delta_\infty> = f(\infty).$$

(19)

Because $f$ is continuous, $\{f(t)\}$ converges with $t$ to $f(\infty)$, implying that $<f, \delta_t>$ converges to $<f, \delta_\infty>$, as required for weak convergence of $\delta_t$ to $\delta_\infty$.

Alooglu’s theorem implies that any bounded net of payoffs of finite portfolio strategies contains a convergent subnet. It is assumed here that the limit point is unique. It follows that all infinite portfolio strategies that are of interest have well-defined payoffs.

An agent implements a *Ponzi scheme* by borrowing one dollar and rolling over the debt forever. Formally, a Ponzi scheme is an infinite portfolio strategy, implying that its dividend is defined as the limit of $-\delta_t$ as $t$ goes to infinity. This, as just discussed, is $-\delta_\infty$. Exactly as with the doubling strategy, the Ponzi scheme may or may not be an arbitrage depending on how one specifies preferences: if they do not depend on dividends at infinity, then the Ponzi scheme is an arbitrage, and as such it must be ruled out via trading restrictions. In that case valuation and utility are discontinuous: $\delta_t$ converges to $\delta_\infty$, but the value of $\delta_t$ does not converge to that of $\delta_\infty$, since the latter is zero. If, on the other hand, preferences do depend on payoffs at infinity, then the Ponzi scheme is not an arbitrage, and there is no necessary obligation to rule it out. If one allows utility to depend on $\delta_\infty$, it is natural to define the utility of $\delta_\infty$ to be the limit of the utilities of the $\delta_t$, so that utility is continuous. In that case valuation is also continuous.

In discussing the doubling strategy we strongly suggested that the discontinuous valuation implied by the received treatment was implausible: why should one treat the dividend of the finitely repeated doubling strategy as getting worse and worse, while simultaneously treating the payoff of the infinitely repeated doubling strategy as an arbitrage? In the present context, in contrast, there does not seem to be an equal presumption against discontinuity of valuation. An agent who has a debt which can be rolled over forever might view that debt as almost equivalent to a debt that has to be repaid at some distant future time. However, it is at least equally plausible to assume that he views the debt as an obligation he never has to repay. In that case even if one defines infinite portfolio strategies as involving dividends at infinity, one is defining utility functions so that Ponzi schemes are arbitrages. We conclude that there may be no simple answer to the question of whether it makes sense to assume that dividends at infinity are or are not valued; it may be that the appropriate specification depends on the context.
2.2 The Stochastic Case

The analysis of the preceding section is easily generalized to stochastic settings. Instead of taking dividends to be measures on \( N_\infty \), we will take them to be measures on \( N_\infty \times N_\infty \), where the first coordinate is time and the second coordinate is the state. We define a metric \( d \) on \( N_\infty \times N_\infty \) by

\[
d((t, \omega), (t', \omega')) = \sqrt{(2^{-t} - 2^{-t'})^2 + (2^{-\omega} - 2^{-\omega'})^2},
\]

where \((0, 0) < (t, \omega) \leq (t', \omega') \leq (\infty, \infty)\). This metric coincides with Euclidean distance if, as above, one associates \( 1, 2, 3, \ldots \) with \( 1/2, 3/4, 7/8, \ldots \) for both \( t \) and \( \omega \), and associates \( \infty \) with 1. Under the topology associated with the distance measure just defined \( N_\infty \times N_\infty \) is compact.

Let \( C \) be the space of continuous real-valued functions on \( N_\infty \times N_\infty \). Under the topology on \( N_\infty \times N_\infty \) generated by the metric (20), \( C \) consists of the functions \( f : N_\infty \times N_\infty \to \mathbb{R} \) such that \( \lim_{\omega \to \infty} f(t, \omega) = f(t, \infty) \), \( \lim_{t \to \infty} f(t, \omega) = f(\infty, \omega) \), and \( \lim_{t, \omega \to \infty} f(t, \omega) = f(\infty, \infty) \), for \( t, \omega \in N \). The point of this specification is to ensure that high values of \( t \) and \( \omega \) are close to \( \infty \). As in the deterministic case, the idea is to enable us to identify measures at the date and state \( \infty \) as dividends of infinite portfolio strategies, and to situate them as limits of payoffs on finite portfolio strategies.

Dividends of finite portfolio strategies can be represented by the elements of \( M(N_\infty \times N_\infty) \) that are zero after some finite date and state, and dividends of infinite portfolio strategies are the weak limits of dividends of finite portfolio strategies. The filtration on \( N_\infty \times N_\infty \) defined in Subsection 1.2 applies in the present setting, with the provisos that \( e^i \) is assumed to include \( \infty \) as well as \( \{t + 1, t + 2, \ldots t + i, \ldots\} \) for finite \( i \) and that agents are assumed to know \( \omega \) at date \( \infty \).

As noted, the dividend of an infinite portfolio strategy will be represented as the weak limit of the payoffs of the associated sequence of dividends of finite portfolio strategies. Most simply, consider the sequence of dividends of finite portfolio strategies represented by the Dirac measures \( \delta_{t,e} \) defined by

\[
\delta_{t,e}(A) = \begin{cases} 
1 & \text{if } t, e \in A \\
0 & \text{if } t, e \notin A 
\end{cases}
\]

for \( e \) an event at \( t \) (i.e., \( e \subset \{e_1, e_2, \ldots e_t, e^i\} \)). Here \( \delta_{t,e} \) indicates one unit of output delivered at date \( t \) and event \( e \), 0 otherwise. It is seen that the space of payoffs of finite portfolio strategies is the span of the \( \delta_{t,e} \) for \( t, e \in N \).

As a special case, \( \delta_{\infty,\infty} \) denotes one unit of output delivered at date \( \infty \) and state \( \infty \). Imposition of the weak topology on \( N_\infty \times N_\infty \) implies that as \( n \) approaches \( \infty \), \( \{\delta_{n,n}\} \) converges to \( \delta_{\infty,\infty} \). To see this, take any test function \( f \) from \( C \). Then

---

3 This section draws heavily on Fisher and Gilles [7].
4 For economists, the best source for the mathematics used in this paper is Aliprantis and Border [1].
Because $f$ is continuous, $\{f(n, n)\}$ converges to $f(\infty, \infty)$, implying that $\langle f, \delta_{n,n}\rangle$ converges to $\langle f, \delta_{\infty,\infty}\rangle$, as required for weak convergence of $\delta_{n,n}$ to $\delta_{\infty,\infty}$.

(22)

Dividends on finite portfolio strategies can be interpreted equally well as functions on $N \times N$ or as measures. To establish the correspondence we define a particular measure $\nu$ as the numeraire, where for each measurable event $e$ at date $t$, $\nu_{t,e}$ is defined as the probability of $e$ occurring. Then the payoff of any finite portfolio strategy interpreted as a measure can be derived by applying its payoff defined as a function as specified above to the numeraire measure $\nu$. Here the dividend interpreted as a function is recognized as the Radon-Nikodym derivative with respect to $\nu$ of the payoff interpreted as a measure. Given that we are assuming risk-neutral valuation, multiplying by probabilities implies that payoffs at particular events are measured by their date-0 values.\(^5\)

As an example, consider the doubling strategy terminated at date 2. Interpreted as a function, its payoff is 1 for $(t, \omega) = (1, 1)$, 1 for $t, \omega = (2, 2)$ and $-3$ for $t, \omega = (2, e^2)$. Interpreted as a measure $\mu_2$ this payoff written

$$\mu_2 = \frac{1}{2}\delta_{1,1} + \frac{1}{4}\delta_{2,2} - \frac{3}{4}\delta_{2,e^2}. \tag{23}$$

Correspondingly, the payoff of the doubling strategy terminated at date $T$ is

$$\mu_T = \sum_{t=1}^{T} 2^{-t}\delta_{t,t} - (1 - 2^{-T})\delta_{T,e^2}. \tag{24}$$

The payoff of an infinite portfolio strategy is defined as the weak limit of the payoffs of the finite portfolio strategies defined by truncating at date $T$, for an increasing sequence of $T$. For the doubling strategy we have

$$\mu_\infty = \sum_{t=1}^{\infty} 2^{-t}\delta_{t,t} - \delta_{\infty,\infty}. \tag{25}$$

Thus the doubling strategy has a (probability-adjusted) payoff of $2^{-t}$ at $t$ for finite $t$, and a payoff of $-1$ at $\infty$ for $\omega = \infty$. Note that the coefficients sum to zero, the initial cost of the doubling strategy. It is seen that the possibility of losing forever is a component of the payoff of the infinitely repeated doubling strategy even though this event occurs with probability zero.

\(^5\)We could have adopted the same numeraire choice in defining portfolio payoffs in the preceding section. Instead we defined payoffs there in terms of contemporaneous value. That choice had the convenient implication that the payoff of the finitely repeated doubling strategy consists of two values (when nonzero). Here, in contrast, the doubling strategy terminated at date $T$ takes on $T+1$ values. The present specification has the attractive property that the payoff of the doubling strategy when unsuccessful converges to $-1$ rather than $-\infty$. 

12
Observe that for the infinitely repeated doubling strategy \( \mu_\infty \) cannot be constructed directly by applying a Radon-Nikodym derivative to \( \nu \). Since \( \nu \) takes value 0 at \((\infty, \infty)\) while \( \mu_\infty \neq 0 \) at \((\infty, \infty)\), \( \mu_\infty \) and \( \nu \) are not equivalent as measures.

We see that extension to the stochastic case involves no surprises, and the same is true of the application to the doubling strategy. Whether or not the doubling strategy involves arbitrage depends on whether utility is defined to depend on components at the date and state \( \infty \). There is nothing to add to the discussion above of how this specification is to be handled.

3 Bubbles

The fundamental value of the payoff of a portfolio strategy is the summed value of its finite-date components. The bubble value is the difference between its initial cost and the fundamental value. If attention is restricted to portfolio strategies without bubbles there is no material difference between portfolio payoffs as conventionally modeled and portfolio payoffs as modeled here: in the latter case the values at the termination date of finite portfolio strategies converge to zero, so that the payoff of infinite portfolio strategies has a zero component at infinity. The discussion of this paper becomes relevant only if one wishes to consider portfolio strategies with bubbles.

Modeling bubble payoffs as weak limits of the terminal payoffs of finite-date portfolio strategies, as proposed here, is not a new idea (see Gilles [8], Gilles and LeRoy [9], [10], Fisher and Gilles [7], Werner [16]). With the exception of Fisher and Gilles [7], those papers did not compactify the payoff index set. As a result, in those papers the limiting payoff took the form of a finitely additive measure. That is a major disadvantage: these are nonconstructive and, as a result, extremely abstract and difficult to work with. Because they depend for their existence on the Axiom of Choice, they are controversial, at least to some mathematicians. As we have just shown, compactifying the payoff index set and representing payoffs by measures makes possible a representation of the limiting payoff that is just as simple as the representation of finite-date payoffs.

4 Conclusion

The simplest—and least controversial—contribution of this paper is to raise questions about the easy characterization of the infinitely repeated doubling strategy as an arbitrage that has to be ruled out by imposing trading restrictions. On the contrary, that conclusion emerges as a consequence of assumptions and modeling conventions the suitability of which is far from obvious. From an analytical point of view this conclusion is good news. If agents are strictly risk averse (and have weakly continuous preferences) our analysis implies that they will avoid the doubling strategy even if it is
available. Therefore there is no need to impose portfolio restrictions to exclude these strategies. Analytically this conclusion is convenient: without trading restrictions choice sets are linear spaces and valuation is linear; with trading restrictions matters are more complicated, even in settings that would otherwise be easy to analyze (see, for example, LeRoy and Werner [13], Ch. 4 and 7).

At a somewhat more controversial level, the analysis here gives a compact—in both senses of the word—analytical vehicle for bubbles. Many analysts point out that there exist theoretical conditions having to do with failure of transversality conditions (see, for example, Santos and Woodford [15] and Huang and Werner [11]) that exclude bubbles. These conditions are sometimes represented as reasonable restrictions. Blanchard and Fischer [2] and others (such as me [12]) have expressed the opposite viewpoint: that these restrictions are farfetched.

There exists some empirical evidence—relying on data prior to the current financial crisis—to the effect that bubbles do not occur. As usual, this evidence is not conclusive, particularly in view of the financial events of the past several years, and in any case it only tests for the existence of the simplest sort of bubbles. We take the view that any strong conclusion in this area is premature. Many phenomena occur in financial markets that are difficult to reconcile with the simplest rational-agent model: one thinks of asset price volatility, the equity premium puzzle, the periodic occurrence of liquidity crises. These phenomena do not appear to produce obvious profit opportunities that agents are irrationally ignoring. Accordingly, it should be possible to analyze them using the orthodox methods of financial economics. If some or all of these phenomena turn out to be connected to bubbles, the analytical techniques developed here may be useful.

References


