Compactifying the Payoff Space: Applications in Financial Economics

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Abstract

Finance models with a finite number of states and dates have properties that do not extend to their infinite counterparts. It is proposed to deal with this situation by appending a date or state called $\infty$ to the payoff index set, and defining a topology such that the payoff index set is compact. This allows a simplified mathematical treatment of a number of topics that are unwieldy when modeled in a setting where the payoff index set is noncompact. Topics discussed include price system bubbles, payoff bubbles, the doubling strategy and non-equivalent martingale measures.

Finance models incorporating a finite number of dates and states have a number of attractive properties not shared by their infinite-dimensional counterparts. For example:

- The initial cost of any finite portfolio strategy (that is, any portfolio strategy that involves nonzero transactions at only a finite number of events) equals the discounted value of its payoff. If infinite portfolio strategies are admitted this may no longer be true. The classic example is the Ponzi scheme, in which an investor borrows money and rolls over the indebtedness forever. The payoff of the Ponzi scheme is zero, yet its initial cost, equal to the initial borrowing, is negative.

- Excluding frictions (specifically, assuming that a firm’s reinvested retained earnings generate returns at the same rate as its original capital, and that this return coincides with the factor at which dividends are discounted), dividend policy does not affect firm value. This is so because lower future dividends imply

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1This paper draws heavily on Fisher and Gilles [3]. I am indebted to Mark Fisher for many conversations about this material. Thanks to Martin Weitzman for comments, and to Yongli Zhang for correcting an error in an earlier draft.
higher retained earnings, and therefore a higher growth rate of invested capital. Assuming that invested capital generates returns at the same rate as dividend streams are discounted, the two effects cancel (Miller and Modigliani [8]). Of course, this argument requires that firms pay a liquidating dividend at the terminal date. With an infinite future this argument fails because there is no terminal date. In the extreme case in which firms never pay a dividend, for example, the firm value justified by dividends is apparently zero. Therefore the dividend-irrelevance proposition is apparently valid only if some dividend strategies are ruled out.

• In infinite settings there appears to exist arbitrage in infinite portfolio strategies even in settings that do not allow arbitrage in finite portfolio strategies. The Ponzi scheme is one example. The classic example is the doubling strategy, in which a gambler who is able to double his bets forever can apparently earn a certain profit.

The loss of these attractive properties of finite models as one goes to infinity in the more general setting is difficult to interpret. What is the source of these qualitative changes—apparent discontinuities—as one moves to the more general setting? It turns out that they are consequences of the way infinity is handled under sequential equilibrium, the equilibrium concept that is usually adopted without explicit discussion or justification.

We propose an equilibrium concept that is an alternative to sequential equilibrium. We call this alternative “Arrow-Debreu equilibrium” because it is more in the spirit of received static Walrasian equilibrium than is sequential equilibrium. Under Arrow-Debreu equilibrium payoffs of infinite portfolio strategies are treated as limits of payoffs of finite portfolio strategies. It turns out that this specification makes it possible to avoid the discontinuities implicit in sequential equilibrium.

There remains the question of how one is to choose between these alternative characterizations of equilibrium in infinite economies. Sometimes one encounters the opinion that specifying sequential equilibrium is more realistic than Arrow-Debreu equilibrium. Since the two equilibrium concepts agree when portfolio strategies are assumed finite, this opinion depends on the presumption that we have reliable intuition to guide us in modeling infinite portfolio strategies. It is far from clear where such intuition comes from; one would like to see the argument in favor of sequential markets made explicit. Any specification involving an infinite number of trades is inherently abstract and unrealistic, and one could take the view that we have no grounds for preferring either definition other than analytical convenience. The criterion of analytical convenience favors the Arrow-Debreu definition, precisely because it avoids the anomalies listed above. The question of which equilibrium concept is to be preferred is best pursued in the setting of specific examples, so we defer further discussion until more context is available.
If the payoffs of infinite portfolio strategies are to be modeled as limits of payoffs of finite portfolio strategies, the setting must be such that these limits exist for a wide range of portfolio strategies. This involves specifying a payoff index set that is compact. The natural numbers are not compact (in the topology to be specified). We remedy this problem by appending $\infty$ to the natural numbers, so that we are allowing portfolio payoffs that have nonzero components at $\infty$. When payoffs and price systems are constructed as function spaces based on the compactified natural numbers, they turn out to have desirable properties. In particular, the anomalies just discussed disappear.

1 Arrow-Debreu and Sequential Equilibrium

As just noted, in multidate models there are two alternative equilibrium concepts: Arrow-Debreu equilibrium and sequential equilibrium. Under Arrow-Debreu equilibrium agents choose optimal intertemporal consumption paths and portfolio strategies subject to an integrated budget constraint, while under sequential equilibrium a separate budget constraint obtains at each date (or, in the stochastic case, at each event). In the finite case the definition of the payoff of a portfolio strategy is the same under both equilibrium concepts: the payoff at any node equals the portfolio chosen at the predecessor node valued at current security prices, plus dividends, less the value of the continuation portfolio.

A basic result of Arrow [2] is that in the finite case equilibria under Arrow-Debreu markets and sequential markets coincide: it does not matter whether agents trade contingent claims before time begins, as under the Arrow-Debreu definition, or trade in real time on actual markets, as under the sequential definition.

The distinction between Arrow-Debreu and sequential equilibrium acquires substance in models that incorporate infinite portfolio strategies. In that case portfolio payoffs are defined as extensions of the operator that characterizes payoffs of finite portfolio strategies. However, there are more than one way to define such extensions. One possible extension involves simply interpreting the definition given above as applying directly to infinite portfolios as well as finite portfolios. Doing so results in a version of the sequential markets concept that applies under infinite portfolio strategies. As noted in the introduction, applying the sequential markets definition of payoffs results in the apparent discontinuities listed there. In particular, in the absence of trading restrictions agents can implement Ponzi schemes—trading schemes that have agents borrowing money and rolling over the debt forever. Under the definition of portfolio payoffs just proposed, a Ponzi scheme involves a negative initial cost at the initial date and a zero payoff at every subsequent date. This is so because the indebtedness implied by the initial borrowing never appears as a component of the portfolio payoff: at each date the investor discharges that indebtedness with the proceeds of further borrowing.

Ponzi schemes, of course, are arbitrages. If optimal portfolios are to exist, the
analyst must impose trading restrictions so as to rule out Ponzi schemes. A variety of possible trading restrictions can be imposed, and these different specifications result in different equilibrium (see, for example, Huang and Werner [5]). That being so, there does not exist a canonical version of sequential equilibrium under infinite trading that corresponds to the standard Walrasian equilibrium that applies in the finite case.

Under the Arrow-Debreu equilibrium concept portfolio strategies are considered as a whole, rather than period-by-period as under sequential equilibrium. The payoff of an infinite portfolio strategy is defined as the limit of the payoffs of the associated finite portfolio strategies (that is, if \( h \) is an infinite portfolio strategy and \( h_n \) is the same portfolio strategy with the positions after date \( n \) set to zero, then the payoff of \( h \) is defined to be the limit of \( \{h_n\} \). If the limit does not exist, the portfolio strategy is undefined. The Arrow-Debreu equilibrium concept requires the analyst to commit to a particular topology, so that the limiting payoff is well-defined. Also, the payoff space must be a Banach space, so that the limiting payoff is representable as a payoff.

It turns out that the transactions that constitute a Ponzi scheme under sequential markets do not create an arbitrage under the Arrow-Debreu equilibrium concept. Accordingly, there is no need to rule out such transactions by imposing transactions costs, although one can still impose such restrictions.

In the following section we set out an analytical framework that allows implementation of the Arrow-Debreu equilibrium concept as we have characterized it. In the substance of the paper we return to the examples listed in the introduction. We show that the Arrow-Debreu equilibrium concept allows a more satisfactory treatment of these puzzles than sequential equilibrium.

\section{Two Function Spaces on \( N_\infty \)}

We restrict attention to a countable payoff index set, which is therefore representable by \( N_\infty = N \cup \{\infty\} \), where \( N \) denotes the natural numbers. We define a metric \( d \) on \( N_\infty \) by

\[
d(i, j) = 2^{-i} - 2^{-j},
\]

where \( 0 < i < j < \infty \), and

\[
d(i, \infty) = 2^{-i},
\]

\( i < \infty \). This metric is indicated if one associates \( 1, 2, 3, \ldots \) with \( 1/2, 3/4, 7/8, \ldots \), and associates \( \infty \) with \( 1 \). Under the topology associated with the distance measure just defined, \( \infty \) is the limit point of the sequence consisting of the natural numbers, so adding \( \infty \) to \( N \) amounts to compactifying \( N \).

A basis for the topology associated with the distance measure defined by (1)-(2) consists of (1) the singleton subsets of \( N \), plus (2) the union of \( \infty \) and the cofinite
subsets of $N$. Thus every open set that includes $\infty$ also includes the tail of $N$ beyond some $n \in N$. In this sense $\infty$ is indistinguishable from the very large elements of $N$. We will see below that this choice of topology will ensure that infinite economies are limits of finite economies.

Let $C$ be the space of continuous real-valued functions on $N_\infty$. Under the topology on $N_\infty$ just specified, $C$ consists of the functions $f : N_\infty \to \mathbb{R}$ such that (1) $\lim_{n \to \infty} f(n)$ exists, and (2) $\lim_{n \to \infty} f(n) = f(\infty)$. We equip $C$ with the topology induced by the norm $\|f\|_\infty = \sup_{n \in N_\infty} |f(n)|$, $n \in N_\infty$.

Consider also $M(N_\infty)$, the space of finite signed measures on the Borel $\sigma$-field $B(N_\infty)$, and define on $M(N_\infty)$ the topology of weak-* convergence, so that $\mu_n \to \mu$ if and only if $\int f d\mu_n \to \int f d\mu$ for all $f \in C$. Then $C$ and $M$ are topological duals under

$$
\langle f, \mu \rangle = \int_{N_\infty} f d\mu = \sum_{i=1}^{\infty} f_i \mu(\{i\}) + f_\infty \mu(\{\infty\})
$$

for $f \in C$, $\mu \in M$. In the applications that follow, $N_\infty$ will be interpreted variously as the set of possible dates and as a probability space. Accordingly, the interpretations of $C$ and $M$ will differ depending on the context.

## 3 Ponzi Schemes

In this section we take the payoff index set $N_\infty$ as representing time, implying that there is no uncertainty. Thus adding $\infty$ to the natural numbers in defining the index set implies that we are formally modeling payoffs in the infinite future and assigning prices to these payoffs.

It turns out that Ponzi schemes are not arbitrages under the Arrow-Debreu definition of equilibrium. To see this, we refer to the function spaces defined in the preceding section. We specify that payoffs are represented as elements of $M$ (i.e., as measures on $N_\infty$).\footnote{For economists, the best source for the mathematics used in this paper is Aliprantis and Border [1].} For example, the Dirac measure $\delta_t$ defined by

$$
\delta_t(A) = \begin{cases} 
1 & \text{if } t \in A \\
0 & \text{if } t \notin A
\end{cases}
$$

for $A \subset N_\infty$ denotes one unit of output delivered at date $t$. As a special case, $\delta_\infty$ denotes one unit of output delivered at date $\infty$.

Now consider the portfolio strategy consisting of borrowing $\$1$ at date $0$, investing it at the risk-free rate for $n$ periods and paying off the loan at date $n$. This is, of

\footnote{Examples of papers that use measures to represent commodities are MasColell [7] and Jones [6].}
course, the finite counterpart of the Ponzi scheme. The initial cost of this portfolio strategy is \(-\delta_0\), as under sequential markets, and its payoff measured in units of date-0 consumption is \(-\delta_n\). The infinitely-repeated version of this portfolio strategy has payoff \(-\lim_{n \to \infty} \delta_n\). Imposition of the weak-* topology on \(N_\infty\) implies that as \(n\) approaches \(\infty\), \(-\delta_n\) converges to \(-\delta_\infty\). To see this, take any arbitrary test function \(f\) from \(C\). Then

\[
<f, \delta_n> = f(n) \text{ and } <f, \delta_\infty> = f(\infty). \tag{5}
\]

Because \(f\) is continuous, \(f(n)\) converges to \(f(\infty)\), implying that \(<f, \delta_n>\) converges to \(<f, \delta_\infty>\), as required for weak-* convergence of \(\delta_n\) to \(\delta_\infty\). Incidentally, it is worth noting that \(\delta_n\) does not converge to \(\delta_\infty\) under the total variation norm: we have \(\|\delta_n - \delta_\infty\| = 2\) for all \(n < \infty\).

We see that the portfolio strategy that produces an arbitrage in the form of a Ponzi scheme under sequential markets does not produce an arbitrage under Arrow-Debreu markets. This is so because a loan that is paid back at date \(n\) in the finite case is paid back at \(\infty\) in the limit. That being so, there is no need to include trading restrictions to make possible existence of optimal portfolios.

Under what we have called the Arrow-Debreu definition of payoffs, agents are treated as paying off (or receiving) at \(\infty\) any value that remains in a portfolio strategy after the finite dates. In the case of the example the negative payoff at \(\infty\) is quantitatively equal (in units of date-0 value) to the date-0 initial cost of the portfolio strategy. In contrast to the case for sequential markets, under Arrow-Debreu markets the value of the payoff of any portfolio strategy, finite or infinite, equals the initial cost of the portfolio strategy.

Because the limit is taken under units of date-0 value, the Arrow-Debreu definition of payoffs as we have defined them automatically imposes continuity of valuation. This is so because the value of any payoff is the initial cost of the portfolio that generates that payoff, and this cost is the same for all portfolios \(h_n\) that are truncated versions of some portfolio strategy \(h\). Because the payoff of \(h\) is the limit of the payoffs of the \(h_n\), and \(h\) is defined to have the same initial cost as the \(h_n\), we have continuous valuation.

For portfolio strategies to which the present value formula applies, the sequential and Arrow-Debreu definitions of payoffs are substantively the same. In those cases the portfolios \(h_n\) have payoffs consisting of dividends up to date \(n\) and a terminal value. As \(n\) rises the terminal value (measured in units of date-0 consumption) converges to zero, by the definition of present value. Therefore adding \(\infty\) to the payoff index set changes nothing, since the indicated portfolio payoff under the Arrow-Debreu definition has zero in this place. Thus the payoff implied by the Arrow-Debreu definition does not differ materially from the payoff as defined under sequential markets. It is only in portfolios like the Ponzi scheme, in which the limit does not converge to zero, that the two definitions differ.

Choosing between these two payoff definitions is, of course, a substantive modeling
judgment. The Arrow-Debreu definition implies that agents treat money that is borrowed with no specific term as having to be paid back at some time in the future, possibly at $\infty$. It enforces this by requiring that any value not paid out in finite dates is paid out at $\infty$. In contrast, sequential equilibrium implies that, in the absence of trading restrictions, this sum never has to be paid (or is never received). Certainly the Arrow-Debreu formalization is more convenient analytically, since it dispenses with the requirement for trading restrictions.

### 4 Dividend Irrelevance

In the preceding example the elements of $C$, the dual space of $M$, were used to test convergence, not to represent price systems. Here we make explicit that price systems are modeled as measures, just like payoffs. For example, in the application to follow the relevant price system involves discounting at a constant rate $\rho$, with $\rho > 0$. This price system is represented by the measure $\lambda$ on $\mathbb{N}_\infty$ defined by

$$\lambda(A) = \sum_{i \in A} (1 + \rho)^{-i}, \quad A \subset \mathbb{N}_\infty.$$  

(6)

It is understood here that $\lambda(\{\infty\}) = 0$, as one would expect from that fact that $(1 + \rho)^{-\infty} = 0$.

Consider a firm that starts with one unit of capital which generates earnings at constant rate $\rho$ at each date. It pays out a proportion $\gamma$ of earnings as a dividend, using retained earnings to acquire more capital. The newly acquired capital also generates earnings at constant rate $\rho$. It is easily verified that the firm pays dividend $z_\gamma(t)$ at date $t$, given by

$$z_\gamma(t) = \gamma \rho (1 + \rho (1 - \gamma))^{t-1}.$$  

(7)

The discounted dividend stream has typical element

$$(1 + \rho)^{-t} z_\gamma(t) = (1 + \rho)^{-t} \gamma \rho (1 + \rho (1 - \gamma))^{t-1},$$  

(8)

and the value of the sequence $\{z_\gamma(t)\}$ for any $\gamma > 0$ is

$$\sum_{t=1}^{\infty} (1 + \rho)^{-t} z_\gamma(t) = 1,$$  

(9)

so that for any $\gamma > 0$ dividend policy does not affect the value of the firm (Miller and Modigliani [8]).

What happens when $\gamma = 0$? From (7) it is clear that for each $t$, $z_\gamma(t)$ equals 0, and the sum of an infinite number of 0s is 0. The apparent conclusion is that the dividend-irrelevance proposition does not apply when $\gamma$ equals zero (Miller and Modigliani [8], note 12). This is a troublesome finding: where did the firm’s value go? However,
the argument is far from water-tight. The payoff stream when $\gamma = 0$ coincides with the limit as $\gamma$ goes to 0 of the payoff stream for positive $\gamma$, where the limit is taken pointwise. The fact that valuation is discontinuous when limits are taken pointwise raises questions about the appropriateness of pointwise convergence, and therefore with the definition of portfolio payoffs as specified in sequential equilibrium.

The analysis leads in a different direction if we recast the firm model just outlined in the analytical framework of Section 2, which relies on weak-* limits rather than pointwise limits. The first step is to redefine the discounted dividend stream as a measure. To that end, define the measure $\mu_\gamma$ by

$$\mu_\gamma(A) = \sum_{t \in A} (1 + \rho)^{-t} \gamma \rho (1 + \rho (1 - \gamma))^{t-1},$$

for $A \subset N_\infty$. The discounted dividend stream as written in the usual form—that is, as characterized in (8)—is recognized as the derivative of $\mu_\gamma$ with respect to the counting measure. Because $\mu_\gamma$ measures dividends in units of date-0 value, the relevant price system is the sequence which assigns 1 to every element of $N_\infty$, including $\infty$. Therefore the value of $\mu_\gamma$ is given by $< 1, \mu_\gamma >$. This equals 1 for each $\gamma > 0$.

The obvious next step is to define the payoff of the "zero"-dividend firm as the weak-* limit of $\{\mu_\gamma\}$ as $\gamma$ approaches zero. Because for $\gamma$ near zero most of the value of the dividend stream occurs in the tail of the dividend sequence—that is, because we have

$$\lim_{\gamma \to 0} < 1, \mu_\gamma(\{T, \infty\}) > = 1$$

for any finite $T$, it follows that $\mu_\gamma$ converges to $\delta_\infty$. Thus the dividend stream generated by the "zero"-dividend firm is one unit of output (measured in value terms) at $\infty$. This payoff has value $< 1, \delta_\infty > = 1$, which equals $< 1, \mu_\gamma >$ for any $\gamma > 0$. Because valuation is weak-* continuous, it is seen that modeling the payoff of the "zero"-dividend firm as the weak-* limit of the payoff of the $\gamma$-dividend firm implies that the dividend-irrelevance proposition applies for $\gamma = 0$ as well as for non-limiting values of $\gamma$.

The applicability of this analysis requires that one accept the maintained assumption that valuation is weak-* continuous, since it is this assumption that allows us to extend valuation from the payoffs of finite portfolio strategies to payoffs with nonzero components at $\infty$. Standard existence theory suggests that valuation will be weak-* continuous if the same is assumed true of preferences. This might or might not be an acceptable assumption. Whether or not one accepts this assumption, however, the present exercise has the important implication that the standard analysis of the "zero"-dividend firm is surely faulty. Under the sequential equilibrium concept the payoff of the zero-dividend firm is the pointwise limit of the payoffs of the $\gamma$-dividend firm. Thus the presumption implicit in sequential equilibrium is that pointwise convergence gives the appropriate definition of convergence. In view of the
fact that virtually none of the standard utility functions are pointwise continuous on infinite-dimensional spaces, there seems little support for this presumption.

In substantive terms, the analysis just presented mirrors exactly that of Gilles and LeRoy [4]. In formal terms, the two analyses differ in that Gilles-LeRoy did not compactify the payoff index set. As a result, the limiting payoff in Gilles-LeRoy’s model took the form of a pure charge (a set function which, unlike a measure, is not countably additive). The fact that we can represent the limiting payoff here as a measure constitutes a major advantage in terms of simplicity: the existence of pure charges requires appeal to the axiom of choice (since proving the existence of pure charges requires the axiom of choice). The fact that pure charges on $\mathbb{N}$ are inherently nonconstructive makes them difficult to work with, compared to measures on $\mathbb{N}_\infty$, which are constructive and easy to work with, as we have seen.

Gilles-LeRoy noted that in their setting the Yosida-Hewitt theorem implies that each element of the payoff space—that is, each charge—can be decomposed into the sum of a measure and a pure charge. They interpreted the measure component of the payoff stream as the fundamental payoff and the pure charge component as a bubble. An analogous interpretation is available here: any measure on $\mathbb{N}_\infty$ can be decomposed into the sum of a measure that is absolutely continuous with respect to the counting measure and another measure that gives weight to $\infty$ alone. The former can be identified as the fundamental component of the payoff and the latter as the bubble component. In terms of the example, the dividend streams associated with $\gamma > 0$ are fundamental payoffs, while the dividend associated with $\gamma = 0$ is a pure bubble. It is worth noting that, as the example indicates, a net of payoffs each element of which has a nonzero fundamental and a zero bubble can converge to a payoff that has zero fundamental and nonzero bubble.

5 The Doubling Strategy

The third example involves uncertainty in an essential way. To model uncertainty in a dynamic setting we interpret $\mathbb{N}_\infty$ as a probability space and define a filtration that represents agents’ information at each date. This, of course, is standard practice in the theory of stochastic processes.

As indicated in the introduction, the doubling strategy is a gambling strategy that involves betting $1 on red on a roulette wheel. Upon winning, the gambler stops betting. Upon losing, the gambler doubles the bet, and continues to do so until he wins. In infinite time, the gambler will win with probability 1, since red will eventually occur except on a set of measure zero. The doubling strategy is generally treated as an arbitrage. Analysts take the view that it is necessary to rule out this trading strategy by invoking trading restrictions (for example, see the discussion in Pliska [9]).

The doubling strategy is essentially a stochastic version of the Ponzi scheme. To see this, it is simplest to recast the roulette wheel as a securities market. Specifically,
suppose that at each date there exists a risky asset with price 1 that generates a payoff of 2 or 0 with equal probability, and a riskless asset with a net interest rate of 0. Then agents can play the untruncated doubling strategy only if they can borrow without limit—that is, only if they can operate a Ponzi scheme. Of course, the question arises why the agent would bother with the doubling strategy if he could operate a Ponzi scheme.

It is clear that, given the analysis above of the Ponzi scheme, the doubling strategy introduces nothing new. We discuss it primarily to demonstrate the application of the Arrow-Debreu equilibrium concept in a stochastic setting. Also, the doubling strategy provides a convenient context in which to make the point that sequential equilibrium involves some implicit presumptions that are, at best, questionable. Analyzing the doubling strategy as a sequential equilibrium presumes that real-world agents—either gamblers or investors—believe that they could manufacture certain wealth from each other by engaging in an infinite number of gambles, each of which is fair for each agent, if only they were able to do so. There is no reason to suppose that anyone is so foolish as to believe this, so why are we creating models that imply it?

Under the Arrow-Debreu equilibrium concept, on the other hand, the possibility of losing is a determinant of the final payoff even though this is a zero-probability event. Assuming that utility is weak-* continuous, so that it assigns utility to the payoffs of infinite portfolio strategies equal to the limit of the utilities of the corresponding finite portfolio strategies, the doubling strategy, far from being an arbitrage, is an extremely unattractive portfolio strategy under most utility functions that incorporate strict risk aversion. Therefore, as with Ponzi schemes, there is no need to impose trading restrictions so as to render the doubling strategy infeasible. Surely this is a more reasonable analysis than that implied by sequential equilibrium.

We now set out the specifics of the analysis of the doubling strategy under the Arrow-Debreu equilibrium concept. First note that, significantly, under the sequential equilibrium concept there does not exist a state corresponding to the event of red never occurring. Under the Arrow-Debreu equilibrium concept this event cannot be neglected even though it occurs with probability zero. Therefore we augment the probability space to allow for an infinite series of black spins. As above, we model the payoffs of infinite portfolio strategies— in particular, the payoff of the doubling strategy—as weak-* limits of the payoffs of the corresponding finitely-repeated portfolio strategies. In the case of a fair roulette wheel, to which we restrict our attention, this limit always exists and takes a very simple form, as we will see.\(^4\)

We take the state of the economy to be the date at which red first occurs. Doing so implies that the state space can be taken to be \(N_\infty\), as in the preceding example, with the element \(\infty\) of \(N_\infty\) denoting the event that red never occurs. We define a

\(^4\)In the general case, so that the house takes a nonzero percentage of the bets (or, in the case of securities, if securities prices reflect risk aversion), the limit may not exist. In that case the interpretation is that the doubling strategy does not exist as a well-defined infinite-time portfolio strategy.
filtration $\mathcal{F}$, which consists of the $\sigma$-fields $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\infty)$. Here $\mathcal{F}_0$ is the trivial partition, $\mathcal{F}_\infty$ is based on the power set of $N_\infty$, and $\mathcal{F}_n$ is based on the partition $\left\{ \{1\}, \{2\}, \ldots, \{n\}, \{ \bigcup_{r=n+1}^\infty \{\tau\} \cup \{\infty\} \} \right\}$. The interpretation is that at date $n$ the agent knows whether or not red has occurred by date $n$, and, if so, at what date this happened. By extension, at date $\infty$ the agent is treated as knowing either the date at which red first occurred, or the fact that it never occurred.\(^5\)

To model the payoff of the doubling strategy, we first specify the measure $\pi$ representing probabilities as

$$\pi(\{\tau\}) = \begin{cases} 2^{-\tau}, & \tau < \infty \\ 0, & \tau = \infty \end{cases}.$$  \hspace{1cm} (12)

Second, we specify the payoff $a_n$ of the finitely-repeated doubling strategy: after playing $n$ rounds of the doubling strategy the payoff is

$$a_n(\{\tau\}) = \begin{cases} 1, & \tau \leq n \\ 1 - 2^n, & n < \tau \end{cases}.$$  \hspace{1cm} (13)

a random variable on $N$. This specification of the payoff of the finitely-repeated doubling strategy amounts to assuming that the roulette wheel is fair (observe that $E(a_n) = 0$, where the expectation is taken with respect to $\pi$). Our evaluation of the limiting payoff of the doubling strategy will be seen below to depend on this assumption.

The next step is to find an expression for $\mu_n$, the payoff of the finitely-repeated doubling strategy expressed as a measure. Define a measure $\nu_n$ on $\mathcal{F}_\infty$ as the product of $a_n$ and $\pi$:

$$\nu_n(\{\tau\}) = a_n(\{\tau\})\pi(\{\tau\}) = \begin{cases} 2^{-\tau}, & \tau \leq n \\ 2^{-\tau}(1 - 2^n), & n < \tau < \infty \\ 0, & \tau = \infty \end{cases}.$$  \hspace{1cm} (14)

The measure $\mu_n$ associated with playing $n$ rounds of the doubling strategy is the restriction of $\nu_n$ to $\mathcal{F}_n$. This is

$$\mu_n(\{\tau\}) = \begin{cases} 2^{-\tau}, & \tau \leq n \\ 2^{-n} - 1, & \tau > n \end{cases},$$  \hspace{1cm} (15)

reflecting the fact that

$$\sum_{\tau=n+1}^\infty 2^{-\tau}(1 - 2^n) = 2^{-n} - 1.$$  \hspace{1cm} (16)

\(^5\)Payoffs that are measurable only with respect to $\mathcal{F}_\infty$ involve a considerable element of abstraction. Rejecting the assumption that there exist payoffs that are measurable with respect to $\mathcal{F}_\infty$ but nonmeasurable with respect to $\mathcal{F}_n$ for finite $n$, however, is tantamount to assuming that agents cannot play the doubling strategy an infinite number of times. This defines away the problem.
Now we are able to identify the limiting payoff $\mu_\infty$ of the infinitely-repeated doubling strategy, expressed as a measure. As expected from the preceding sections, we define this as just the weak-* limit of $\mu_n$ as $n$ goes to $\infty$:

$$
\mu_\infty = \lim_{n \to \infty} \mu_n = \begin{cases} 2^{-\tau}, & \tau < \infty \\ -1, & \tau = \infty \end{cases}
$$

(17)

This expression for $\mu_\infty$ is intuitively plausible inasmuch as it coincides with (15) except that the term $2^{-n}$ disappears in the limit. The expression may, of course, be justified formally by taking the typical continuous function $f$ on $N_\infty$ as a test function for weak-* convergence.

The payoff of the doubling strategy played $n$ times can be represented equivalently by the measure $\mu_n$, given by (15), or the random variable $a_n$, given by (13). The point of this exercise is to demonstrate that the former expression is more convenient when it comes to characterizing the limiting payoff. This is so because the payoff of $\mu_n$ if $\tau > n$ converges to $-1$, reflecting the fact that $\mu_n$ weak-* converges to $\mu_\infty$. In contrast, the payoff of $a_n$ converges to $-\infty$ if $\tau > n$, implying that $a_n$ does not converge (in, for example, mean-square) to any random variable as $n$ goes to $\infty$. Thus (13) does not apply if $n = \infty$. Nonexistence of a limiting density, of course, reflects the fact that $\mu_\infty$ is not absolutely continuous with respect to probability, since $\{\infty\}$ is a null set for probability, but not for $\mu_\infty$. Here we see the advantage of working with the underlying measures $\mu_n$: we have convergence as $n$ goes to $\infty$ instead of nonconvergence.

A complete treatment of the economic theory of the doubling strategy would involve defining the utility of payoffs represented as measures. In an expected utility context this would involve extending expected utility from the space of payoffs that have derivatives with respect to probability to the larger space that includes limit points. We do not pursue this line here. It is worth observing, however, that the conclusion of earlier sections, that pointwise convergence does not provide a sensible way to define limiting payoffs, applies here also.

If one uses weak-* convergence, as here, rather than pointwise convergence to define limiting payoffs, the demonstration that the doubling strategy generates an arbitrage fails. This is clear from (17): $\mu_\infty$ is not a positive measure. This is a sensible conclusion: there is no economic or mathematical justification for the presumption, implicit in the sequential definition of equilibrium, that the decreasing probability of a loss as $n$ goes to $\infty$ somehow trumps the size of the loss in the limit. If weak-* convergence is used to define the limiting payoffs, it follows that excluding arbitrage does not imply the necessity of ruling out the doubling strategy. This is, of course, exactly the same conclusion as applied to Ponzi schemes.

We may compare the present analysis with that of Gilles and LeRoy [4]. These authors, using $N$ as the payoff index set rather than $N_\infty$, showed that the weak-* limit of the payoff on the finitely repeated doubling strategy is a charge which has a nonzero pure charge component. In contrast, compactifying the payoff index set
here allows us to avoid introducing pure charges.

6 Conclusion

In many papers the author presumes that the definitions of payoffs and budget constraints specified in sequential equilibrium is just as appropriate when the setting incorporates infinite portfolio strategies as in the finite case. Santos and Woodford [10] is one example. We question this presumption: sequential equilibrium has implications that may or may not be acceptable. We have proposed an alternative characterization of equilibrium, one closer to the Walrasian paradigm, and have argued that this alternative framework produces a very different analysis—and, in our view, an analysis that is much more sensible—in cases where the two paradigms differ.

References


