Infinite Portfolio Strategies

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Abstract

In infinite-date models the received definitions of the payoffs of finite portfolio strategies imply discontinuous valuation. Accordingly, in the absence of trading restrictions, arbitrage results when infinite trading strategies are admitted. We propose an alternative that is free of these problems. The alternative produces a cleaner, if more abstract, treatment of equilibrium in financial models in infinite-date settings. We consider the bearing of the revised treatment on the theory of rational bubbles, overlapping generations models and equivalent martingale measures.

1 Introduction

In finite settings there is no ambiguity about the definition of portfolio payoffs: at any event the payoff of a given portfolio strategy equals the value of the portfolio chosen at the immediately preceding event, plus dividends, minus the value of the portfolio chosen at that event. In infinite settings this definition of payoffs may not be applicable, even in the discrete-time setting examined in this paper, since events may not have immediate predecessors. Even when it is applicable its appropriateness is not as clear. For example, under the definition just presented the payoff of a portfolio strategy that is self-financing at every date—that is, where the payoff is zero at every date—is zero. Where did the value represented by the initial cost of the portfolio go?

Assume that time is infinite and countable, and therefore that it can be indexed by the natural numbers. Under this specification the setting assumes an infinite

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1I have received helpful comments from Stephen A. Clark, Mark Fisher, Christian Gilles, Zari Rachev, Jan Werner and Bill Zame. I am indebted to seminar participants at University of Karlsruhe, University of Trento, University of Rome, the University of Southern California and at the Southwest Economic Theory Conference for comments.

2We choose a discrete-time setting so that the theory corresponds directly to an extended discrete-time example, presented in Subsection 8.1.
horizon if there exists no terminal date, or a finite horizon if there exists a terminal date $T$. We will assume that in the infinite-horizon case each date other than the initial date has an immediate predecessor, whereas in the finite-horizon case the terminal date has no immediate predecessor. In the finite horizon case time may be interpreted as $0, 1/2, 3/4, 7/8, \ldots$, for example, in which case the horizon occurs at date $T = 1$.

The answer to the question “Where did the value go?” is different in the two cases as usually modeled. In the infinite-horizon case the value is pushed off into the indefinite future. In the finite-horizon case, however, it is assumed that the portfolio is liquidated at $T$, with the payoff at $T$ being defined as the pointwise limit of the portfolio values at $t$ for $t \to T$.

We propose two modifications of this treatment. First, we adopt a symmetric treatment of the two cases by appending a date called “$\infty$” in the infinite-horizon case and defining the payoff at infinity of an infinite portfolio as the limit of the value of the portfolio as $t$ goes to infinity. In other words, we treat $\infty$ in the infinite-horizon case the same as $T$ in the finite-horizon case. Second, in the case of uncertainty we replace pointwise limits with weak* limits. These changes will be seen to allow a more intuitive treatment of several major issues in infinite-time finance: bubbles, equilibrium in overlapping generations models, and existence of equivalent martingale measures.

2 The Deterministic Case

The proposed formal treatment of infinite portfolio strategies is most easily illustrated in a deterministic setting, although we will see that some of the important analytical issues disappear in that case. A finite portfolio strategy is represented by a finitely nonzero sequence $\Theta = \{\theta_t\}$, where $\theta_t$ is an $n$-tuple of security holdings at date $t$ ($t = 0, 1, \ldots$). The payoff $X = \{x^\Theta_t\}$ of a finite portfolio strategy is defined at all dates $t$ (other than the initial date 0 and, in the finite-horizon case, events at $T$) by

$$x^\Theta_t = [\pi_t + d_t] \theta_{t-1} - \pi_t \theta_t,$$

$t = 1, 2, \ldots$, where $\pi_t$ is an $n$-tuple of security prices at date $t$ and $d_t$ is an $n$-tuple of dividends at time $t$. The payoff at the initial date 0 is zero. The initial cost of $\Theta$ is $\pi_0 \theta_0$.

The foregoing description of finite portfolio strategies applies in both the finite-horizon case and the infinite-horizon case. Characterization of payoffs of infinite portfolio strategies requires some preliminary setup. In the absence of arbitrage in finite portfolio strategies, there exists a state price deflator $M$, a strictly positive

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3There are other possibilities. For example, an alternative version of the finite-horizon case might be specified so that each date has an immediate predecessor (in that case time is indexed by $\ldots - 1, 0, 1, 2, \ldots, T - 1, T$). These are minor variations of the cases analyzed.
process such that the initial cost \( \pi_0 \theta_0 \) of any finite portfolio strategy \( \Theta \) equals its summed payoff valued using \( M \):

\[
\pi_0 \theta_0 = \sum_t x_t m_t,
\]

where \( m_t \) is the date-\( t \) component of \( M \).

Let \( l^p \) be the set of (generally infinitely nonzero) sequences \( X \) for which \( \|X\|_p \), defined by

\[
\|X\|_p = \left[ \sum_t |x_t|^p \right]^{1/p},
\]

is finite. Following Fisher and Gilles [12], suppose that \( M \in l^p \) for some \( p \), and consider \( \Theta^q \), the set of (generally infinitely nonzero) portfolio strategies \( \Theta \) that are mapped by (1) to elements of \( l^q \), where

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

Since \( M \in l^p \) and, for any \( \Theta \in \Theta^q \), \( X \in l^q \), it follows that \( Y = MX \in l^1 \), implying that the image \( X \) of \( \Theta \in \Theta^q \) under (1) has finite date-0 value. Here \( Y \) can be interpreted as resulting from applying a numeraire change to \( X \), so as to measure in units of date-0 value. Note that we are not characterizing \( X \) or \( Y \) as the payoff of \( \Theta \), except when \( \Theta \) is a finite portfolio strategy.

In the finite-horizon case, let \( v_t \) be the date-\( t \) value of portfolio strategy \( \Theta \in \Theta^q \), measured in units of date-0 value (i.e., \( v_t = \pi_t \theta_t m_t \)), and let \( v_T \) be the limit of \( v_t \) as \( t \) approaches \( T \), assuming the limit exists. Payoffs of infinite portfolio strategies \( \Theta \) are denoted \((Y, v_T)\) whenever the limit exists. The first term represents the component of the payoff that occurs at finite dates, while the second term is the component of the payoff that occurs at \( T \). For example, consider a self-financing portfolio strategy with initial cost 1. If terminated at date \( t \), this portfolio strategy has payoff of 1 (measured in units of date-0 value) at date \( t \)—the liquidating dividend—and zero elsewhere. In terms of the notation for payoffs just set out, this is \((1_t, 0)\), where \( 1_t \) is the sequence consisting of 1 at \( t \), zero elsewhere. If instead this portfolio strategy is rolled over at each date, so that it has zero dividend at each finite \( t \), then the payoff consists of \((0, 1)\), where the first term is the zero sequence and the second term consists of a liquidating dividend of 1 at \( T \).

When the horizon is finite it appears natural to define the payoff of an infinite portfolio strategy at a finite horizon in this way—that is, as a liquidating dividend equal to the limiting value of the portfolio strategy. However, when the horizon is infinite, so that time is indexed by \((0,1,2,...)\), established practice is to identify

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In order to represent payoffs of finite portfolio strategies as a special case of payoffs of infinite portfolio strategies, one would replace \( X = \{x^\Theta_t\} \), where \( x^\Theta_t \) is defined by (1), by \((X,0)\), and similarly with \( Y \).
the payoff with $Y$ alone, so that the second term in $(Y, v_T)$ is deleted (Santos and Woodford [38], for example). Thus the payoff of a portfolio strategy that is self-financing at every date is the sequence 0. This allows arbitrage in infinite portfolio strategies even when there is no arbitrage in finite portfolio strategies. The standard example is the Ponzi scheme, which consists of borrowing 1 and rolling over the debt forever, resulting in a zero payoff. Because such arbitrages can be operated on any scale, their existence is inconsistent with equilibrium when agents have increasing preferences. Such trading strategies must be ruled out via trading restrictions. These trading restrictions considerably complicate formal analysis.

As indicated in the introduction, the proposal here is to treat the infinite-horizon case the same as the finite-horizon case. Thus we append a date called $\infty$ and define payoffs of portfolio strategies to be of the form $(Y, v_\infty)$, the infinite-horizon analogue of $(Y, v_T)$ in the finite-horizon case. As in the finite-horizon case, $v_\infty$ is defined to be the limit of the $v_t$.

Our proposal, to define payoffs of portfolio strategies as $(Y, v_\infty)$ or $(Y, v_T)$ rather than $Y$, is purely a matter of definition. Carrying around the additional terms $v_\infty$ or $v_T$ has no consequences by itself, since any operation that can be defined on $Y$ can obviously be defined on $(Y, v_\infty)$ or $(Y, v_T)$. However, the alteration in the definition of payoffs suggests changes that are not so innocuous: for example, it is natural to define an arbitrage as a portfolio with nonpositive initial cost and positive $(Y, v_\infty)$ or $(Y, v_T)$ (not all zero), rather than just positive $Y$ as is generally done. Under this redefinition of arbitrage, the absence of arbitrage in finite portfolio strategies (which was assumed in the construction of $Y$) implies that there cannot exist arbitrage in infinite portfolio strategies. From $v_t = y_{t+1} + v_{t+1}$ it follows that the initial cost of any infinite portfolio strategy equals $\sum_t y_t + v_\infty$. Therefore the initial cost of any portfolio strategy for which $\{y_t\}$ and $v_\infty$ are positive and not all equal to zero is strictly positive. In the case of the Ponzi scheme, the portfolio strategy of borrowing money and rolling over the loan forever is impossible: our formal definition of the payoff of this portfolio strategy entails representing the loan as necessarily being repaid “at infinity”. Thus the payoff of the Ponzi scheme with initial value $-1$ is $(0, -1)$.

3 The Stochastic Case

For the most part the definition of payoffs of infinite portfolio strategies in the stochastic case follows the same lines as in the deterministic case outlined in the preceding section. However, we will see that a substantive issue arises in defining the limiting payoff of an infinite portfolio: in what sense is the limit to be taken? In the deterministic case payoffs are scalars, so the issue does not arise. However, when they are random variables the issue is nontrivial. The usual specification is that the limit is taken pointwise on the set of states, and in this section we follow this precedent. In the following section we show that this specification leads to discontinuous valuation
of finite portfolio strategies, and therefore to arbitrage in infinite portfolio strategies.

Uncertainty is represented by a probability space \((\Omega, \mathcal{F}, \mu)\), where \(\mathcal{F}\) is a \(\sigma\)-algebra on \(\Omega\) and \(\mu\) is a probability measure on \(\mathcal{F}\), and an event tree with a finite number of events (subsets of \(\Omega\)) at each date. There is a single event \(\xi_0\) at date 0. As in the deterministic case, agents trade \(n\) securities at each node of the event tree, where \(n\) is finite. As above, finite portfolio strategies are representable as \(\Theta = \{\theta(\xi)\}\), where \(\theta(\xi)\) is an \(n\)-tuple in which the \(i\)-th element represents the holding of security \(i\) at event \(\xi\).

The payoff \(X = \{x_\Theta(\xi)\}\) of a finite portfolio strategy is defined at all events \(\xi\) (other than the initial event and, in the finite-horizon case, events at \(T\)) by

\[
x_\Theta(\xi) = [\pi(\xi) + d(\xi)]\theta(\xi^-) - \pi(\xi)\theta(\xi),
\]

the stochastic analogue of (1). Here \(\xi^-\) is the event that immediately precedes \(\xi\) and \(\pi(\xi)\) and \(d(\xi)\) are \(n\)-tuples of security prices and dividends at event \(\xi\). The payoff at the initial event \(\xi_0\) and, in the finite-horizon case, payoffs at \(T\) are zero. The initial cost of \(\Theta\) is \(\pi(\xi_0)\theta(\xi_0)\), written below as \(\pi_{\xi_0}\). These specifications are the obvious uncertainty counterparts of the corresponding definitions in the deterministic case.

Portfolio strategies representable by infinitely nonzero sequences are infinite portfolio strategies. As in the deterministic case we will adopt a symmetric treatment of the finite-horizon and infinite-horizon cases by appending a date called \(\infty\) in the latter case and defining the payoff at \(\infty\) in the same manner as the payoff at \(T\) in the finite-horizon case.

Suppose that \(M \in L^p\) for some \(p\), and consider \(\Theta^q\), the set of (generally infinitely nonzero) portfolio strategies \(\Theta\) that are mapped by (5) to elements of \(L^q\), where again

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

\[\text{We will generally use capital letters to denote stochastic processes and lower-case letters to denote random variables, n-tuples or scalars.}\]
Since $M \in \mathcal{L}^p$ and $X \in \mathcal{L}^q$, it follows that $Y = MX \in \mathcal{L}^1$, implying that the image $X$ of $\Theta \in \Theta^q$ under (5) has finite value under (7).

Let $v_t$, a random variable on $\Omega$, be the date-$t$ value of portfolio strategy $\Theta \in \Theta^q$, measured in units of date-0 value (i.e., $v_t = \pi_t \theta_t m_t$), and let $\tilde{v}_\infty$ be the pointwise (on $\Omega$) limit of $v_t$ as $t$ approaches $\infty$, assuming the limit exists. Specification of the pointwise limit follows established practice in the continuous-time finance literature (Duffie [8], Appendix D, for example).

Payoffs of infinite portfolio strategies $\Theta$ are denoted $(Y, \tilde{v}_\infty)$ in the infinite horizon case, and $(Y, \tilde{T})$ in the finite-horizon case, whenever the limit exists. The first term represents the component of the payoff that occurs at finite dates, while the second term is the component of the payoff that occurs at infinity (or at $T$).

$\text{3.1 Martingales and Closed Martingales}$

Define $L^p$ as the space of random variables $v$ with finite norm, where the norm is defined by

$$\|v\|_p = \left[ E |v|^p \right]^{1/p}. $$

A stochastic process $V = \{v_t\}$ defined on $(0, 1, 2, \ldots)$ is a martingale if

$$v_t \in L^1 $$

all $t$, and

$$v_t = E_t(v_\tau) $$

for $t \leq \tau$. The martingale convergence theorem (see, for example, Chung and Williams [5]) implies that if $V$ is a bounded martingale, there exists a random variable $\tilde{v}_\infty$ that is the pointwise limit of the $v_t$. If $\tilde{v}_\infty$ has the same expectation as the other elements of $V$, then $\{v_1, v_2, \ldots, \tilde{v}_\infty\}$ is a martingale. In that case we will call $V$ a closed martingale, with $\tilde{v}_\infty$ the closing term. This terminology is drawn from Tucker [41].\(^6\)

A self-financing infinite portfolio strategy is an infinite portfolio strategy for which $Y = 0$, so that its payoff measured in date-0 units equals $(0, \tilde{v}_\infty)$ for some random variable $\tilde{v}_\infty$ (hereafter the exposition assumes an infinite-horizon setting for convenience; for the finite-horizon case just substitute $T$ for $\infty$). For a self-financing portfolio strategy, $V = \{v_t\}$ is a martingale. Therefore the martingale convergence theorem guarantees the existence of $\tilde{v}_\infty = \lim_{t \to \infty}(v_t)$ as an element of $L^1$.

$\text{4 Discontinuous Valuation}$

The martingale convergence theorem makes clear that $\tilde{v}_\infty$ does not necessarily have the same expectation as the $v_t$. Therefore $V$ may not be a closed martingale. Under\(^6\)I am indebted to Stephen A. Clark for suggesting the term and pointing out this reference.

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the additional restriction that \( V \) is uniformly integrable (or if any of a variety of other conditions is satisfied) then \( E(\hat{v}_\infty) = E(v_t) \) for all \( t \), so that \( V \) is a closed martingale, with \( \hat{v}_\infty \) the closing term. In that case \( E(\hat{v}_\infty) \), the date-0 expectation of the limiting payoff, equals the initial cost of the infinite portfolio strategy with payoff \((0, \hat{v}_\infty)\).

However, the uniform integrability condition may not be satisfied. If it is not, the \( v_t \) may converge (pointwise almost everywhere) to zero even for portfolio strategies with nonzero initial cost, since when uniform integrability fails the expectation of the limit is not necessarily equal to the limit of the expectations. In that case, valuation of finite portfolio strategies is pointwise discontinuous. Similarly, expected utility is generally pointwise discontinuous (this was noted by Omberg [33]; see also Ingersoll [25]).

Defining payoffs of infinite portfolio strategies as limits implies that arbitrage exists when infinite portfolio strategies are considered and valuation of finite portfolio strategies is discontinuous. To see this, suppose that \( \theta \) and \( \theta' \) are self-financing infinite portfolio strategies with date-\( t \) values \( v_t \) and \( v'_t \). Because \( \{v_t\} \) and \( \{v'_t\} \) are martingales, we have that \( E(v_t) \) takes on the same value for all \( t \), and similarly for \( E(v'_t) \). Assume that \( \lim_{t \to \infty} v_t = \lim_{t \to \infty} v'_t \) almost everywhere, and also \( E(v_t) \neq E(v'_t) \); the definition of continuity implies that there exist portfolio strategies with this property if valuation is discontinuous. Then the infinite portfolio strategy \( \theta - \theta' \) has payoff \((0, 0)\) but nonzero initial cost. Therefore the law of one price fails, and this portfolio strategy or its negative is an arbitrage.

4.1 Example

The doubling strategy and the “suicide strategy” (as Harrison and Pliska [23] called the negative of the doubling strategy) are familiar examples of the failure of the law of one price that results from discontinuous valuation. Suppose that at each date \( t \) an agent can buy without cost \( \theta_t \) shares of a gamble each share of which produces a payoff of \(+1\) or \(-1\) with equal probability. The trading strategy is to keep playing, doubling the bet at each stage, until the first win occurs, at which time the agent stops betting. After \( t \) rounds the value \( v_t \) of the doubling strategy is

\[
v_t = \begin{cases} 
1 & \text{with probability } 1 - 2^{-t} \\
1 - 2^t & \text{with probability } 2^{-t}.
\end{cases}
\] (12)

The expectation of this payoff is, of course, zero, as it must be because the initial cost of the portfolio strategy is zero and the gamble in each round is fair. However, the payoff converges pointwise to 1 almost surely, an arbitrage. Note here that \( V \) is not uniformly integrable.

A uniform lower bound on wealth rules out the doubling strategy (Dybvig and Huang [10]), but permits a suicide strategy, in which an agent gambles until his first loss rather than his first win. This strategy produces a loss almost everywhere.
4.2 Critique

Defining the payoffs of infinite-horizon portfolio strategies as pointwise limits makes mathematical sense, but it is not clear that doing so makes economic sense. In the case of the doubling strategy, the prospect of not winning after \( t \) rounds contributes \( 2^{-t} - 1 \) (the product of \( 1 - 2^t \) and \( 2^{-t} \)) to the expected payoff of the finite doubling strategy. This converges to \(-1\). Despite this, the prospect of never winning contributes nothing to the expected payoff of the infinite-horizon doubling strategy. Thus the fact that the probability of never winning goes to zero trumps the fact that the payoff in the event of not winning goes to \(-\infty\), this despite the fact that the product of the two converges to \(-1\).

Agents are modeled as believing that the opportunity to gamble with each other forever enables each of them to produce wealth out of nothing. Do we want to build that much irrationality into the definition of portfolio payoffs?

In a curious passage discussing the doubling strategy, Delbaen and Schachermayer ([7], p. 465), observed that the player who doubles his bet until the first time he wins .... has an almost sure win .... However, his accumulated losses are not bounded below. Everybody, especially the casino boss, knows that this is a very risky way of winning 1 ECU.

They then argue from this that the doubling strategy has to be ruled out via trading restrictions. What is curious about the passage is that under the received mathematical treatment—defining the limiting payoff via pointwise convergence—the strategy, like any arbitrage, is modeled as in fact having no risk of loss, and it is (assumed to be) attractive to risk-averse investors precisely for that reason. Rather than motivating imposition of trading restrictions, Delbaen and Schachermayer, apparently without realizing it, were in fact expressing doubts about the received mathematical treatment, which represents as riskless the payoff of a gambling strategy that, according to them, is far from riskless.

When the analysis involves an infinite number of states rather than dates, most economists understand that notions of convergence are very much at issue. For example, issues similar to those discussed here arise in the Arbitrage Pricing Theory of Ross [35], [36]: analysts assume absence of “approximate arbitrage” so as to avoid introducing arbitrage when defining payoffs of diversified portfolios—usually implicitly, it is true—as limits of payoffs of finite portfolios. They (sometimes) state explicitly that by absence of approximate arbitrage they mean continuity in the mean-square norm. Nevertheless, when the commodity index set represents time rather than states of nature, “convergence” always seems to mean “pointwise convergence”. This is so despite the inconvenient fact that, under this specification, valuation necessarily involves discontinuity, which economists find difficult to motivate in other contexts.
In the absence of trading restrictions, existence of arbitrage implies nonexistence of optimal portfolios (when agents prefer more to less), so it cannot occur in equilibrium. A variety of trading restrictions may be used to create an environment in which agents have optimal portfolios. These restrictions complicate the analysis of valuation to different degrees and in different ways.\textsuperscript{7}

5 Payoffs as Weak* Limits

The difficulties discussed above—discontinuous valuation of finite portfolio strategies, implying the existence of arbitrage when payoffs of infinite portfolio strategies are modeled as limits of payoffs of finite portfolio strategies—are all consequences of working with pointwise limits. The principal argument in favor of pointwise convergence is that it is tractable, although why that argument appears decisive as regards time but not uncertainty is unclear, as discussed in Subsection 4.2.\textsuperscript{8}

The values \(v_t\) of self-financing infinite portfolio strategies lie in \(L^1\), so it is natural to consider replacing pointwise convergence with convergence in the \(L^1\) norm. However, with this change there is no assurance that the limiting payoff exists. For example, in the case of the doubling strategy considered in Section 4.1, \(\{v_t\}\) diverges in the \(L^1\) norm, implying the unsatisfactory outcome that the doubling strategy is not a well-defined portfolio strategy.

Suppose instead that we embed \(L^1\) in its second topological dual, the space \(ba(\Omega, \mathcal{F}, \mu)\) of signed charges on \(\mathcal{F}\). Assuming that the net \(\{v_t\}\) directed by \(t\) is bounded, Alaoglu’s theorem guarantees the existence of weak*-convergent subnets in \(ba\). We will take the limit points \(v_\infty\) of such nets as the date-\(\infty\) payoffs of infinite portfolio strategies.

The Yosida-Hewitt theorem guarantees that the charge \(v_\infty\) is the sum of a measure and a pure charge. The former can be identified with an element of \(L^1\), by the Radon-Nikodym theorem. Further, the measure component coincides with the pointwise limit \(\tilde{v}_\infty\) of the values \(v_t\) discussed above (this follows from Fisher and Gilles’ \[12\] Theorem A.5). Thus the present treatment differs from that of Section 3 in including

\textsuperscript{7}The consequences of imposing different types of trading restrictions were summarized in the February 17, 2002 version of this paper. See also Huang and Werner [24].

\textsuperscript{8}The remainder of this paper presumes some knowledge of functional analysis. Much of the relevant material is summarized in Gilles [17], Gilles and LeRoy [19], [20] and Fisher and Gilles [12] in a context very close to that of the present paper. For more extended and more rigorous discussion, see Royden [37], Aliprantis and Border [1], Dunford and Schwartz [9] and Bhaskara Rao and Bhaskara Rao [34].

In the discussion to follow, whenever there exists an isometric isomorphism between two function spaces, we will not distinguish between the two. For example, the space of norm-continuous functions on \(L^\infty\) is isometrically isomorphic to a space of charges. Instead of adopting notation that distinguishes between the two and stating the isometry, we will simply identify charges and continuous functions, thereby economizing on verbiage and notation. The sources cited provide the rigorous treatment.
the pure charge component of the limiting payoff. For example, in the context of the doubling strategy, the limiting payoff consists of 1 (that is, the measure associated with the random variable 1) plus a pure charge centered on the event that the agent loses forever.

5.1 Continuous Valuation

As just noted, when the pure charge component of $v_{\infty}$ equals zero, $v_{\infty}$ can be identified with a measure, and this measure in turn is representable as a random variable; further, the integral of the price system with respect to the measure coincides with the expectation of the random variable. In this case the date-0 values $V$ of a self-financing portfolio strategy are representable as a closed martingale, with $v_{\infty}$ as the closing term.

In general the pure charge component of $v_{\infty}$ may be nonzero, implying that $v_{\infty}$ is not a random variable and its expectation is not defined. In that case $V$ is not a closed martingale. However, the date-0 value of $v_{\infty}$ can still be identified with its integral (see Gilles and LeRoy [20], appendix, for a minimally technical discussion of integration with respect to a charge), and this integral coincides with the initial cost of the portfolio strategy.\(^9\)

Weak* convergence of $v_t$ to $v_{\infty}$ plus the fact that $V$ is a martingale means that $E(zv_t)$ for any $t$ equals the integral of $zv_{\infty}$ for any $z \in L^\infty$. Since the functional 1 is an element of $L^\infty$, we have that $E(v_t)$ equals the integral of $v_{\infty}$ for any $t$. It follows that the date-0 value of $v_t$ equals that of $v_{\infty}$ for any $t$, implying that valuation is continuous.

In the context of the doubling strategy the measure component of the limiting payoff has expectation, and therefore date-0 value, equal to 1. The pure charge component has value $-1$. Thus the date-0 value of the limiting payoff—the sum of the values of the measure component and the pure charge component—equals the initial cost (zero) of the portfolio strategy, and also equals $E(v_t)$ for any finite $t$.

Continuous valuation implies that allowing for infinite portfolio strategies does not introduce arbitrage: if there existed an infinite portfolio strategy with positive and nonzero payoff and negative initial cost, continuity implies that there would also exist finite portfolio strategies with the same properties. This, however, was ruled out in assuming the existence of a state price deflator. In particular, the doubling strategy is not an arbitrage: the pure charge component of $v_{\infty}$ is negative.

Nonexistence of arbitrage implies that there is no need to impose portfolio restrictions although, of course, one can still do so if desired. Thus the canonical model of

\(^9\)Note that we cannot generally write the integral of $v_{\infty}$ as $E(v_{\infty})$. This is so because $v_{\infty}$, as a charge, is not a random variable, so its expectation is not defined. It is true that in the special case when $v_{\infty}$ has a zero pure charge component it can be identified with a measure (by the Yosida-Hewitt theorem), and therefore (by the Radon-Nikodym theorem) with a random variable in $L^1$, so in that case there is no difficulty with identifying the date-0 value of $v_{\infty}$ with $E(v_{\infty})$. 

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finite-time-and-state finance, which specifies unrestricted portfolio strategies, has a
direct counterpart in the infinite case. In contrast, we saw that under pointwise con-
vergence trading restrictions must be imposed to prevent arbitrage, and the features
of the model depend critically on what restrictions are adopted. Therefore under
pointwise convergence there can be no canonical model.

In the absence of trading restrictions, portfolio choice sets are linear spaces. Since
with payoffs of infinite portfolio strategies defined as weak* limits there is no need
for trading restrictions, it follows that valuation is linear in the canonical version. In
contrast, when payoffs of infinite portfolio strategies are defined as pointwise limits,
the fact that portfolio restrictions must be imposed implies that in such settings
valuation may or may not be linear depending on the restrictions imposed and whether
or not they are binding in equilibrium (for an example in which the trading restrictions
produce linear valuation, see Magill and Quinzii [30], [31]).

5.2 Utility when Consumption Bundles Are Charges

Having defined portfolio payoffs as charges, it follows that consumption bundles are
charges as well. When these have zero pure charge components the utility of such
consumption bundles can be calculated in the usual way. What about when they
have nonzero pure charge components, as with the doubling strategy? It makes sense
to extend the utility function from $L^1$ to $ba$ by defining the utility of a charge as
the limiting utility of a net of consumption bundles—represented as measures—that
weak*-converges to that charge. In effect this specification involves assuming weak*-con-

In many cases the utility assigned to infinite portfolio strategies will be $-\infty$. For
example, this is true of the doubling strategy for agents who are risk averse (and not
nearly risk neutral in the sense of Fisher and Gilles [13]).

In order to determine the utility of consumption bundles resulting from infinite
portfolio strategies one must use utility functions the support of which includes all
the relevant payoffs. In the case of the doubling strategy, for example, all negative
payoffs must be admissible. Within the class of linear risk tolerance utility functions,
negative exponential utility and quadratic utility have this property, while power and
logarithmic utility do not. It is easily checked that under negative exponential or
quadratic utility the expected utility of the value of the doubling strategy—given by
(12)−converges to $-\infty$\textsuperscript{10}. Far from being a riskless arbitrage, it is seen that the

\textsuperscript{10}Modeling the payoff of the doubling strategy as a limit was proposed by Omberg [33], as noted
above. Omberg also noted that the expected utility of the doubling strategy is $-\infty$ under neg-
ative exponential utility. However, there are difficulties with Omberg’s discussion: it appears to
suggest that the universal practice of defining random variables as equivalent if they differ only a
set of measure zero needs qualification. If so, the consequences for probability theory would be
considerable.

Under the present framework no such qualification is necessary: charges can be centered on events
of probability measure zero, as in the doubling strategy, but even then they are defined on sets of
doubling strategy is very risky, and is very unattractive to risk-averse investors for that reason.

6 Application: Bubbles

The remainder of this paper consists of three applications of the modeling convention for infinite portfolio strategies outlined above.

The preceding discussion has direct implications for the analysis of (rational) bubbles. First, we define the fundamental of any portfolio payoff \((Y, v_\infty)\) as its stochastic process component \(Y\). The remaining component \(v_\infty\) is the bubble. In particular, the fundamental of any self-financing portfolio strategy is zero, so that the nonzero component of its payoff is identified with a bubble. Correspondingly, the fundamental value of \((Y, v_\infty)\) is the date-0 value of the fundamental which, from (7) and \(Y = MX\), equals \(\sum_t E(y_t)\). The bubble value equals the integral of \(v_\infty\). For self-financing portfolio strategies the bubble value equals \(E(v_t)\) for any \(t\).

In the foregoing development it was tacitly assumed that the securities out of which portfolios are composed do not themselves have bubbles. In many applications, such as the analysis below of overlapping generations models, one wants to relax this assumption. This is easily achieved: one can modify the definition of a security’s payoff to coincide with the payoff of the portfolio consisting of a buy-and-hold strategy involving that security.11 The definitions of fundamental value, bubble and bubble value for securities are the obvious counterparts of the corresponding definitions for portfolio strategies.

The definitions just presented agree with general usage in earlier discussions of bubbles, except that in most discussions no distinction is drawn between bubble and bubble value, and similarly for fundamental and fundamental value. Suppression of this distinction in the earlier studies is justified because these studies generally imposed trading restrictions, implying that asset values are not generally related linearly to payoffs. In that case the fundamental and bubble cannot necessarily be identified with distinct parts of the payoffs. Here, however, we do not impose trading restrictions, implying that valuation is linear. Therefore portfolio values always have counterparts in portfolio payoffs, and it is necessary to adopt terminology that distinguishes between portfolio values and portfolio payoffs.

Several recent discussions of bubbles adopt a framework similar to that set out here, but define bubbles differently. For example, Loewenstein and Willard [29] implicitly and Fisher and Gilles [12] explicitly identified bubbles with the pure charge

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11 No circularity is involved because doing this does not imply the need for any alteration in the characterization of portfolio payoffs: portfolio payoffs at \(\infty\) can still be defined as limits of portfolio values.
component of $v_\infty$. In contrast, Clark [6] identified bubbles with the measure component of $v_\infty$. Both of these alternative definitions entail a departure from received usage, which identifies fundamental value via the present-value relation (that is, as the value attributable to payoffs at finite dates). Adhering to received usage, as we have done, implies identifying bubbles with $v_\infty$, rather than either its measure component or its pure charge component separately. This is purely a semantic matter, but it is difficult to see any rationale here for deviating from the received definitions.

Gilles [17] was first to point out the connection between bubbles and charges. Gilles and LeRoy [18], [19], [20], [21] continued work along these lines. All of these papers, however, worked exclusively with valuation operators and payoff spaces, so that there was no representation of portfolio strategies. Specifically, in Gilles [17] and Gilles and LeRoy [19] the payoff space was $L^\infty$ and valuation operators lay in $ba$, while the reverse specification was adopted in Gilles and LeRoy [20]. In the present analysis we follow Fisher and Gilles [12] in representing bubbles as a component of portfolio payoffs measured in units of date-0 value (this was implied in working with $Y = MX$ rather than $X$). Therefore bubbles are not traceable to either the valuation operator or the payoff space separately.

The present framework for the analysis of bubbles is superior to that of the earlier papers. First, it makes explicit the connection between the analysis of bubbles and the characterization of portfolio payoffs (via pointwise or weak* limits). Second, it situates bubbles as consequences of portfolio strategies that push their payoffs into the infinite future, which has no direct connection either with the specification of payoff spaces or of price systems. Making explicit the connection between bubbles and portfolio strategies allows integration of the analysis of bubbles into the mainstream finance literature (for an example in the finance literature of the lack of such integration, see Sethi [40], where it is pointed out that the present-value relation may fail; no connection is made there with the economics literature on bubbles).

7 Application: Overlapping Generations\textsuperscript{12}

We will consider the simplest overlapping generations model: there is an initial date 0, a single good is traded, there is no uncertainty or production, and the initial generation is endowed with intrinsically useless money. Specifically, assume that at each date $t$, $t = 0, 1, 2, \ldots$, a generation of agents is born. This generation is young at date $t$ and old at date $t + 1$, and has an endowment of $e^t_t$ when young and $e^t_{t+1}$ when old. Following the development in Section 2 above, we will measure these endowments, and all quantities, in units of date-0 value (implying that $e^t_t$ and $e^t_{t+1}$ have endogenous components).

\textsuperscript{12}Familiarity with overlapping generations models is assumed. Azariadis [2] has a textbook-level introduction to overlapping generations models; Geanakoplos [14] provided an excellent discussion of the differences between overlapping generations models and Arrow-Debreu models.
The money of overlapping generations models, being intrinsically useless, has a payoff of 0 at finite dates. In equilibria in which money is valued, therefore, the money endowment of generation 0 can be identified with a payoff of \( \{0, \mu\} \) for some \( \mu > 0 \). Thus in modeling money as a bubble, we are assuming that generation 0 has an endowment of \( e_0^0 \) at 0, \( e_1^0 \) at 1 and \( \mu \) at infinity, all measured in units of date-0 consumption. Generation 0’s excess demand functions can be written as \( Z_0^0(e_0^0, e_1^0, \mu) \) at 0, \( Z_1^0(e_0^0, e_1^0, \mu) \) at 1 and \( Z_\infty^0(e_0^0, e_1^0, \mu) \) at infinity. Generation \( t = 1, 2, \ldots \) has excess demand functions \( Z_t^t(e_t^t, e_{t+1}^t) \) at \( t \) and \( Z_{t+1}^t(e_t^t, e_{t+1}^t) \) at \( t + 1 \). These excess demands do not depend on the value \( \mu \) of the monetary endowment because generation \( t \) has no monetary endowment for \( t > 0 \).

In equilibrium the excess demand of generation 0 at 0 is

\[
Z_0^0(e_0^0, e_1^0, \mu) = 0
\]

since agents at date 0 have no one to trade with. Further, we have

\[
Z_\infty^0(e_0^0, e_1^0, \mu) = -\mu
\]

(14)

because generation 0 will sell its endowment of money. This excess demand reflects the constraint that consumption at \( \infty \), as at all dates, is nonnegative, and that generation 0 does not value consumption at \( \infty \) (or at any date other than 0 and 1).

From generation 0’s budget constraint,

\[
Z_0^0(e_0^0, e_1^0, \mu) + Z_1^0(e_0^0, e_1^0, \mu) + Z_\infty^0(e_0^0, e_1^0, \mu) = 0,
\]

(15)

there follows

\[
Z_1^0(e_0^0, e_1^0, \mu) = \mu.
\]

(16)

From this, the market-clearing condition

\[
Z_{t+1}^t(e_t^t, e_{t+1}^t) + Z_{t+1}^{t+1}(e_{t+1}^t, e_{t+2}^{t+1}) = 0
\]

(17)

implies

\[
Z_{t+1}^t(e_t^t, e_{t+1}^t) = \mu, \quad Z_{t+1}^{t+1}(e_{t+1}^t, e_{t+2}^{t+1}) = -\mu,
\]

(18)

\( t = 1, 2, \ldots \).

For example, assume that \( (e_t^t, e_{t+1}^t) = (1, 1/3) \), all \( t \), and that in equilibrium we have \( \mu = 1/3 \). Assume also that the utility function of generation \( t \) is \( \log(e_t^t) + \log(e_{t+1}^t) \). In that case equilibrium consumption equals \( (1, 2/3) \) for generation 0 and \( (2/3, 2/3) \) for later generations. Further, \( (e_t^t, e_{t+1}^t) \) measured in units of date-0 consumption equals \( (e_t^t, e_{t+1}^t) \) measured in natural units, so no renormalization is needed to convert endowments from natural units to units of date-0 value.

Note that in the foregoing discussion the market at infinity does not clear: generation 0 sells a payoff at infinity, but nobody buys this payoff at infinity. It is this sale that provides a positive wealth transfer to generation 0 that is not offset by a
negative wealth transfer from any other generation.\footnote{Walras' Law implies that wealth transfers must sum to zero. However, Walras' law applies only when the aggregate endowment has finite value, which is not the case here. It is this fact that underlies proof of nonexistence of bubbles in (most) finite-agent settings and in “well-behaved” overlapping generations models. For a discussion of overlapping generations models that emphasizes the role of wealth transfers, see Marshall, Sonstelie and Gilles [32].} In this respect our formulation has a strong similarity to that of Geanakoplos [14]. The principle difference is that Geanakoplos, drawing on Geanakoplos and Brown [15], adopted a different formalism. In his setting the relevant market that does not clear is that at date $T$, in a finite version of the model. He then identified $T$ with an infinite number via nonstandard analysis. In the present setting, as just noted, the market that does not clear is that representing the payoff at $\infty$ on the endowment of money, but it amounts to the same argument.

In the setting here there exists a continuum of equilibria indexed by $\mu$. It follows that the dimension of indeterminacy of monetary overlapping generations equilibrium is 1, agreeing with the conclusion of Geanakoplos. We expect that the methods outlined here could be applied to analyze the dimension of indeterminacy of the equilibrium set in models with many goods and many infinite-lived securities (Wilson [42], Kehoe and Levine [26], [27]).

8 Application: Equivalent Martingale Measures

The analysis just presented allows a simple characterization of when equivalent martingale measures exist: the equivalent martingale measure corresponding to any numeraire choice exists if the portfolio strategy associated with that numeraire is a closed martingale, and only then.

Setting up this result requires some preparation. In finite economies there exists a correspondence between probability measures and numeraire choices:\footnote{For an excellent discussion emphasizing the analytical benefits of an appropriate numeraire choice, see Geman, El Karoui and Rochet [16].} for any probability measure $\nu$ that is equivalent to the original measure $\mu$ there exists a positive-valued self-financing portfolio strategy with value $Z$ (and $z_0 = 1$) such that $V/Z$ is a $\nu$-martingale, and conversely, where $V$ is the value of any self-financing portfolio strategy. If the money market account is chosen as the numeraire portfolio strategy $Z$, then $\nu$ is the risk-neutral measure. Dybvig and Ross [11] called this result the “Fundamental Theorem of Finance”.

In finite economies the proof is elementary. We have already seen that if $v_t$ is the date-$t$ value of any self-financing portfolio strategy, then in a finite setting $m_tv_t$ is a $\mu$-martingale:

$$m_tv_t = E_t(m_Tv_T),$$

(19)
where $T$ is the terminal date. Now let $Z$ be the value of a positive-valued self-financing portfolio strategy with initial value 1 that is to be taken as numeraire. Dividing (19) by $m_t z_t$ results in

$$
\frac{v_t}{z_t} = E_t \left( \frac{m_T z_T v_T}{m_t z_t z_T} \right) = E_t^\nu \left( \frac{v_T}{z_T} \right),
$$

if $\nu$ is the measure that has $m_T z_T$ as its Radon-Nikodym derivative. Therefore $V/Z$ is a martingale under $\nu$. Further, the change-of-measure process $m_t z_t = E_t (m_T z_T)$ is a martingale under $\mu$.\(^\text{15}\)

It is known from a large recent literature (for example, Schachermayer [39], Delbaen and Schachermayer [7], Back and Pliska [3], Gilles and LeRoy [21] and the papers cited in these) that neither the necessity nor the sufficiency of this condition extends to infinite settings, at least in the absence of qualification. We give a simple and intuitive account of why the Fundamental Theorem of Finance may fail in an infinite setting.

In infinite time $MZ$ is a martingale, but it is not necessarily a closed martingale. If it is not a closed martingale, then $m_t z_t$ does not converge to a random variable with expectation 1. This is true under both pointwise and weak* convergence: $m_t z_t$ may converge pointwise to a random variable with expectation not equal to 1, in which case $m_t z_t$ weak*-converges to a random variable with expectation not equal to 1, in which case $m_t z_t$ weak*-converges to a random variable (because it has a nonzero pure charge component). Then $MZ$ does not define the Radon-Nikodym derivative associated with a change of measure, implying that no equivalent martingale measure is associated with $Z$.

In particular, there exists a risk-neutral measure if and only if the money-market account is a closed martingale.

These results may be stated equivalently in terms of bubbles. A positive-valued self-financing infinite portfolio strategy is identified with a change of measure if and only if the bubble associated with that portfolio strategy has a zero pure charge component. In particular, there exists a risk-neutral measure if and only if the bubble associated with the money-market account has a zero pure charge component.

8.1 Example

The preceding analysis is illustrated in an example. Consider a representative agent model in which the agent maximizes

$$
\sum_{t=0}^{\infty} 2^{-t} E [\ln(c_t)].
$$

\(^{15}\)Baxter and Rennie [4] has a very accessible and intuitive discussion of changes of measure in the discrete-time case.
The agent’s endowment equals \( 2^t \), \( t = 0, 1, 2, \ldots \) if the state\(^{16}\) is high at \( t \) or at any date prior to \( t \), and 1 otherwise. The transitions between the high and low states are governed by a \( 2 \times 2 \) transition matrix with \( 1/2 \) in each position, so that the states are independent and equally likely. The state at date 0, the initial date, is \( L \).

It is easily checked that event prices at date \( t \)—the prices of one unit of consumption contingent upon a particular sequence of \( H’ \)’s and \( L’ \)’s up to date \( t \)—equal \( 2^{-3t} \) if the state is high at \( t \) or at any date prior to \( t \), and \( 2^{-2t} \) otherwise. Accordingly, the state price deflator \( M \)—the ratio of event prices to probabilities—is given by

\[
m_t = \begin{cases} 2^{-2t} & \text{if } \tau \leq t, \\ 2^{-t} & \text{otherwise,} \end{cases}
\]

where \( \tau \) is the date of the first high endowment realization. Since the endowment realization after \( \tau \) equals \( 2^t \) regardless of the state, it is clear that all uncertainty is resolved at \( \tau \). The event that \( \tau \leq t \) has probability \( 1 - 2^{-t} \) for any \( t \).

Note here that, because of the presence of the discount factor in the utility function (21), the state price deflator declines with \( t \). However, it declines an order of magnitude faster when \( t > \tau \) than otherwise, and this is what gives rise to the distinctive features of the example.

Let \( r_t \) be the gross one-period interest rate from \( t - 1 \) to \( t \). It is given by

\[
r_t = \begin{cases} 4 & \text{if } \tau < t, \\ 4(1 + 2^{-t})^{-1} & \text{otherwise.} \end{cases}
\]

The date-\( t \) value \( b_t \) of the money-market account \( B \), equal to the cumulated value of one unit of consumption invested at the one-period interest rate and rolled over at each date, is

\[
b_t = \prod_{i=1}^{t} r_i = \begin{cases} 2^{2t} \prod_{i=1}^{t} (1 + 2^{-i})^{-1} & \text{if } \tau \leq t, \\ 2^{2t} \prod_{i=1}^{t} (1 + 2^{-i})^{-1} & \text{otherwise.} \end{cases}
\]

Define \( W \) as the product of \( M \) and \( B \). From (22) and (24), \( W \) is given by

\[
w_t \equiv m_t b_t = \begin{cases} \prod_{i=1}^{t} (1 + 2^{-i})^{-1} & \text{if } \tau \leq t, \\ 2^t \prod_{i=1}^{t} (1 + 2^{-i})^{-1} & \text{otherwise.} \end{cases}
\]

The process \( W \) is a martingale.\(^{17}\) However, it is not a closed martingale. To see this, note that the event that \( t \) consecutive realizations of \( L \) occur, which has

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\(^{16}\)Here we are not using the term “state” in its dynamic programming sense, since the level of the aggregate endowment at any date is not a sufficient statistic for the past history of the economy up to that date.

\(^{17}\)For example, the date-0 value of \( W \) is 1, its date-1 values are \( w_{HH} = 2/3 \), \( w_{HL} = 4/3 \), its date-2 values are \( w_{HHH} = w_{HHL} = 2/3 \), \( w_{HLH} = 8/15 \) and \( w_{LLL} = 32/15 \).
probability $2^{-t}$, contributes $\prod_{i=1}^{t} (1 + 2^{-i})^{-1}$ to $E(w_t)$. This term converges to 0.4194, which therefore equals the date-0 value of the pure charge component of the bubble $w_\infty$. Therefore the expectation of the measure component of $w_\infty$ equals 0.5806 ($= 1 - 0.4194$). The portfolio strategy $W$ is seen to be a suicide strategy.

The foregoing discussion establishes that, despite the absence of arbitrage, there does not exist a risk-neutral probability measure in this example.\footnote{As a martingale, $w_t$ equals the date-t value of a self-financing portfolio strategy. The fundamental component of the payoff of $W$ is zero and its bubble component is a charge with date-0 value 1. However, suppose that instead of rolling over the portfolio, the investment is terminated as soon as the first high state occurs. In that case the fundamental component of the payoff of $W$ is 0.5806 and its bubble value is 0.4194, agreeing with Loewenstein and Willard’s [29] terminology. For this portfolio strategy the bubble is a pure charge centered on the event $(L,L,L,...)$.}

## 9 Comparison with Kreps and Fisher-Gilles

Our definition of payoffs of infinite portfolio strategies via limits invites comparison with that of Kreps [28], where topological considerations were first introduced in defining terms related to arbitrage. Kreps began by pointing out that absence of arbitrage is not a sufficient condition for existence of strictly positive valuation operators in infinite-dimensional cases. To address this problem, Kreps defined a free lunch as a sequence of zero-valued portfolio strategies which converge to a strictly positive payoff (here we are simplifying Kreps’ definition in several respects). Absence of free lunches is stronger than absence of arbitrage.

Kreps worked exclusively with finite portfolio strategies, in contrast to the present paper where we are dealing with infinite portfolio strategies. To see this, consider Kreps’ result that if there exist free lunches then no agent with continuous preferences has an optimal portfolio strategy. If there exists a sequence of zero-valued finite portfolio strategies the payoffs of which converge to a strictly positive payoff, then continuity implies that there exist zero-valued finite portfolio strategies the payoffs of which are strictly utility-increasing. This is inconsistent with existence of optimal portfolios. This demonstration does not involve infinite portfolio strategies in any way.

However, Kreps (and also Harrison and Kreps [22]) implicitly defined payoffs of infinite portfolio strategies in discussing the doubling strategy. Like many others, they presumed that a pointwise limit is appropriate, and observed that under that definition the payoff of the doubling strategy is an arbitrage. As noted above, our proposal is to substitute weak* limits for pointwise limits.

Fisher and Gilles [13] suggested instead that payoffs of infinite portfolio strategies be defined by taking limits in the topology under which continuity of agents’ utility is defined (this is not completely explicit in their discussion). This alternative is completely in the spirit of Kreps’ definition of free lunches, but it has some consequences that may not be acceptable. In finite settings the payoffs of portfolio strategies are
defined directly from portfolio strategies (i.e., portfolio weights), security payoffs and security prices. Under our definition the same is true in infinite settings. Under Fisher-Gilles definition, in contrast, preferences are also involved. For example, depending on the topology used to define continuity of preferences, a given infinite portfolio strategy may or may not have a well defined payoff, and if it does, this payoff may or may not be an arbitrage.

Our definition has several attractive properties. For example, we saw that every positive-valued self-financing infinite portfolio strategy has a well-defined payoff. This being so, the connection between such portfolio strategies and equivalent martingale measures that characterizes finite settings carries over to infinite settings (with the proviso that uniform integrability must obtain). Under Fisher-Gilles’ definition, in contrast, the fact that payoffs are not always well defined implies that the connection between portfolio strategies and equivalent martingale measures is lost.

This said, it is worth repeating that our definition of payoffs independently of preferences has the consequence that portfolio strategies may generate payoffs with utility equal to minus infinity. For example, we saw that the doubling strategy has this property when agents are risk averse. This agrees with the implicit usage in such sources as Omberg [33], but there is no denying that such portfolios have no counterpart in finite settings.

10 Conclusion

We have proposed modeling payoffs of infinite portfolio strategies via charges. It may be objected that introducing a mathematical entity that is not even a random variable is unnecessary at best, and patently unrealistic at worst. The charge of lack of realism must be rejected: any model in which agents trade an infinite number of times is already unrealistic, yet one would not dismiss all of continuous-time finance on these grounds. Similarly, the portfolio strategy for duplicating the payoff of an option in the Black-Scholes model involves an infinite volume of trade. This is not viewed as a critical failure, nor should it be.

Whether or not one thinks it is necessary, or fruitful, to introduce charges depends on whether or not the discussions above of bubbles, overlapping generations models and equivalent martingale models seem worthwhile. The reader is the best judge of this.

References


