A Model of Endogenous Cross-Holdings in Oligopoly

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Abstract

A network approach is proposed to analyze endogenous cross-holdings and their competitive implications. The approach is motivated by the bilateral arrangement of passive ownership held by Microsoft in Apple in 1997. For a model of Cournot oligopoly with a homogeneous product, it is shown that (i) there is a unique “pairwise stable” cross-holding structure, in which firms hold the maximum allowable amount of passive ownership in rivals when there are two firms; (ii) there is a wide range of pairwise stable cross-holding structures when there are three firms; and (iii) with four or more firms, pairwise stability implies that none of the firms holds any passive ownership in rivals. An implication of these results is that bilateral formation of cross-holdings by horizontal firms yields more collusive outcomes when the industry is more concentrated.

JEL classification: C72; G0; L1

Keywords: Cross-holding; Cournot equilibrium; Oligopoly, Pairwise stability

1 Introduction

Cross-holding refers to a situation where a firm acquires passive ownership in another firm, which entitles the acquiring firm a share in the acquired firm’s profits but not in

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its decision making. Cross-holdings are not rare in practice. For example, Northwest Airlines acquired a 14% equity position in Continental Airlines in 1998. Microsoft acquired about 7% of the nonvoting stock of Apple in 1997. The Japanese and the U.S. automobile industries are two of many that feature complex webs of cross-holdings (see Alley, 1997).

A well-known result in the literature concerning competitive implications of cross-holdings states that with Cournot competition on the output market, the output market equilibrium will become less competitive, with aggregate output level falling toward the monopoly level as cross-holdings increase. As argued in Reynolds and Snapp (1986), after a firm has entered into a long equity position in a rival firm, it is induced to take into consideration the effect of its own output decision on the rival’s profit. This consideration makes the firm compete less aggressively, because in so doing the firm can increase the profit to the rival and hence its stake in the rival’s profit. Thus, if all firms hold passive ownership in each other, they are both induced to produce less which leads to greater profits for all of them.

The preceding comparative static result takes cross-holdings as given instead of being endogenously determined by the firms. As we illustrate later in this paper, depending on the structure of cross-holdings, two firms may be better off reducing cross-holdings. The reason is because their less competitive behavior may induce the other firms to compete more aggressively. A more complete analysis of the anti-competitiveness of cross-holdings would therefore require that cross-holdings be endogenously determined.

In this paper, we propose a network approach to analyze firms’ incentives for bilateral formation of cross-holdings and their competitive implications. Intuitively, when two firms can increase total profits by a bilateral change of cross-holdings, they can always agree on mutually beneficial prices for making the changes (see, for example, Farrell and Shapiro, 1990, p. 287). Thus, a collection of cross-holdings is pairwise stable if no two firms can increase total profits by any bilateral change of cross-holdings. This notion of pairwise stability is adapted from Jackson and Wolinsky (1996). Jackson and Wolinsky consider networks on the set of players of a game with discrete links, in the sense that the link between two players takes either value 1 (the link is present) or value 0 (the link is absent). Considering a collection of cross-holdings as a (directed) network, the values of the links are continuous. For example, the value of 5% equity shares held by firm i in firm j for the link from i to j is different from the value of 10% equity shares held by i in firm j for the same link. See Jackson (2008) for

\footnote{\textquotedblleft Microsoft Investments Draw Federal Scrutiny,	extquotedblright Pittsburg Post-Gazette, August 10, 1997, B-11; \textquotedblleft Corel Again Buys a 'Victim' of Microsoft Juggernaut,	extquotedblright The Ottawa Citizen, February 8, 200, C1. \textquotedblleft Antitrust Investigations examine Microsoft, Apple Deal,\textquotedblright Journal Record The (Oklahoma City), by Bob Drummond Bloomberg News, August 20, 1998.}

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Our consideration of bilateral formation of cross-holdings is motivated by the bilateral arrangement of passive ownership held by Microsoft in Apple. On August 5, 1997, Apple and Microsoft reached an agreement according to which Microsoft could purchase shares of Apple’s series nonvoting, convertible, preferred stock at the aggregate price of 150 million dollars. The bilaterally arranged purchase price was below the market.

Our model has two stages. The formation of cross-holdings takes place in Stage 1. In Stage 2, firms choose outputs to compete in Cournot fashion taking as given cross-holdings formed in Stage 1. Since cross-holdings are passive, firms cannot condition cross-holdings or their prices on subsequent output choices.

We focus on characterizations of pairwise stable cross-holdings and their anticompetitive implications. To this end, given the non-contingencies of cross-holdings and their prices upon output choices, we leave the determination of the prices for cross-holdings not explicitly modeled because they do not affect firms’ output choices. We require instead that they guarantee that cross-holdings be “individually rational”. Individual rationality in this paper means that no firm is made worse off from participating in cross-holdings than not participating. We show that the individual rationality of a cross-holding structure implies that the total profit of each firm is no less than what it gets in the Cournot equilibrium with no cross-holdings. Thus, the formation of an individually rational cross-holding structure is (weakly) Pareto improving, as compared with the Cournot equilibrium with no cross-holdings.

In reality, the amount of a firm’s equity holding in a rival firm is restricted and cross-holdings do not lead any firm to shutdown production. To capture these empirical facts, an upper bound on the amount (less than 50 percent) is imposed on passive ownerships firms are permitted to hold in rivals, such that firms’ Cournot equilibrium outputs are positive given permissible cross-holdings. In order to focus on firms’ incentives for forming cross-holdings that is driven by the desire to increase market power without any basis for synergy, our analysis will be carried out for a Cournot oligopoly with a linear demand and identical linear cost functions.

Given permissible cross-holdings, the “profits from production” of the firms in Cournot equilibrium on the output market are determined by their shares of the total output based on the cross-holding structure. A change in cross-holdings between two developments and applications of network models.

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firms generates an “output effect” and a “price effect” on their total profits. The price effect of an increase in cross-holdings increases total profits while the output effect decreases total profits. The reason is that an increase in cross-holdings between two firms decreases both their output shares and total equilibrium output. We show that (i) there is a unique pairwise stable cross-holding structure, in which firms hold the maximum permissible amount of passive ownership in rivals when there are two firms; (ii) there is a wide range of pairwise stable cross-holding structures when there are three firms, in each of which one firm holds identical permissible amount of passive ownership in each of the other two firms, which in turn do not hold any passive ownership in their rivals; and (iii) pairwise stability of cross-holding structures implies that none of the firms holds any passive ownership in rivals when there are four or more firms. With two firms, the price effect dominates the output effect regardless of the cross-holding structure. Thus, the firms should increase cross-holdings to the maximum allowable amounts. In contrast, with four or more firms, the output effect dominates the price effect for any two firms regardless of the cross-holding structure. As a result, no two firms have any incentive to engage cross-holdings with each other. The case with three firms is more subtle. The two effects can be balanced at some cross-holding structures, but additional conditions are needed for those cross-holding structures to be pairwise stable.

Our results imply that the bilateral formation of cross-holdings by horizontal firms results in more collusive outcomes when the industry is more concentrated. For the case with four or more firms competing in Cournot fashion on the output market, the competitive implications of pairwise stable cross-holding structures are the same as those based on unilateral stability of cross-holding structures in Flath (1991) and Reitman (1994). For the case with three or less firms, both these latter two papers conclude that no firm unilaterally has incentives to engage cross-holdings. In contrast, there are non-zero pairwise stable cross-holdings. To the extent that cross-holdings between horizontal firms are observed in practice (e.g., cross-holdings between Microsoft and Apple, between Northwest and Continental Airlines, etc.), our results are more consistent with empirical cases.

The rest of the paper is organized as follows. Section 2 introduces the basic model and the concepts of pairwise stability and individual rationality. Section 3 presents the main results and Section 4 concludes. Proofs of results are organized in an appendix.
Consider a market with $n$ identical firms producing a homogeneous good. Market inverse demand is denoted by $P(X)$, where $X$ is the total output. Firm $i$’s technology exhibits constant returns to scale, which results in constant marginal cost of $c > 0$ and zero fixed cost for all $i$. Denote by $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in})$ the vector of equity shares $\alpha_{ij}$ of firm $i$ in firm $j$, and by $\tau_i = (\tau_{i1}, \tau_{i2}, \cdots, \tau_{in})$ with $\tau_{ii} = 0$ the vector of prices $\tau_{ij}$ firm $i$ pays to firm $j$ for cross-holdings. Here, $\alpha_{ii}$ is the amount of firm $i$’s ownership held by its external stockholders (stockholders outside the industry). Let $\alpha$ and $\tau$ denote matrices with the $i$-rows equal to $\alpha_i$ and $\tau_i$, respectively, for all $i$, and let $I$ denote the $n \times n$ identity matrix. In the present setting, $I$ represents the cross-holding structure with no firm holding passive ownership in rivals. Given $\alpha$ and $\tau$, firm $i$’s total profit at output profile $x = (x_1, x_2, \cdots, x_n)$ is given by

$$\pi_i(x, \alpha) + \kappa_i(\alpha, \tau), \quad (1)$$

where

$$\pi_i(x, \alpha) = [P(X) - c]X_i \quad (2)$$

is referred to as firm $i$’s profit from production in the presence of cross-holdings,

$$\kappa_i(\alpha, \tau) = \sum_{j=1}^{n} [\tau_{ji}\alpha_{ji} - \tau_{ij}\alpha_{ij}] \quad (3)$$

as its net payment for cross-holdings, and $X_i = \sum_{j=1}^{n} \alpha_{ij}x_j$ as its equity-based share of the total output $X = \sum_{i=1}^{n} x_i$.\footnote{We follow the formulation of firms’ profits as in Reynolds and Snapp (1986), under which the portion of firm $i$’s operating earnings $(\sum_{j \neq i} \alpha_{ji})[p(X) - c]x_i$ going to competitors is not included in the profit that guides the firm $i$’s output choice (Reynolds and Snapp, 1986, p. 144).} Given $\alpha$, firm $i$’s output choice is affected by $\pi_i(x, \alpha)$ only.

Because of the timing difference between the formation of cross-holdings and output choices, we solve the decision problems via backward induction. Given $\alpha$, we denote Cournot equilibrium by $x^*(\alpha) = (x^*_1(\alpha), \cdots, x^*_n(\alpha))$. Set

$$\pi^*_i(\alpha) = \pi_i(x^*(\alpha), \alpha). \quad (4)$$

This is the reduced form of firm $i$’s profit from production as a function of the cross-holding structure.
2.1 Individual Rationality

Cross-holdings are voluntary. Hence, it is natural to require that no firm be made worse off from engaging in cross-holdings than not engaging in any cross-holding.

**Definition 1** We say that a cross-holding structure represented by matrix $\alpha$ is individually rational if there exists a price matrix $\tau$ such that for all $i$,

$$\pi^*_i(\alpha) + \kappa_i(\alpha, \tau) \geq \min_{\alpha: \alpha_{ij}=\alpha_{ji}=0, \ j\neq i} \pi^*_i(\alpha). \quad (5)$$

The right-hand-side of (5) is the profit level firm $i$ can guarantee itself by not engaging in any cross-holding. We show in the next section that a firm’s individually rational profit level coincides with its profit level in Cournot equilibrium without any cross-holding.

A cross-holding structure with zero payments for cross-holdings may violate individual rationality. For example, let $n = 3$, $p(X) = 1 - X$, and $c = 0$. Consider $0 < \delta < 1/2$ and

$$\alpha = \begin{pmatrix} 1 & \delta & \delta \\ 0 & 1 - \delta & 0 \\ 0 & 0 & 1 - \delta \end{pmatrix}. \quad (6)$$

Simple calculation shows that given cross-holding structure $\alpha$ in (6), the Cournot equilibrium is

$$x_1^*(\alpha) = \frac{1 - 2\delta}{4 - 2\delta} \text{ and } x_2^*(\alpha) = x_3^*(\alpha) = \frac{1}{4 - 2\delta}.$$

By (2) and (4), we have

$$\pi^*_1(\alpha) = \frac{1}{(4 - 2\delta)^2} \text{ and } \pi^*_2(\alpha) = \pi^*_3(\alpha) = \frac{1 - \delta}{(4 - 2\delta)^2}.$$

Should payments for cross-holdings be zero, both firm 2 and firm 3’s profits would be below their Cournot equilibrium profit levels without any cross-holding.

2.2 Pairwise Stability

Cross-holdings between two firms induce them to compete less competitively. However, their less competitive behavior may make the other firms the primary beneficiaries. To see this, let with $n$, $P(X)$, and $c$ be as in the previous example. Then, $\pi^*_i(I) =$
Now, consider cross-holdings between firms 2 and 3 that result in
\[
\alpha = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - \alpha_{32} & \alpha_{23} \\
0 & \alpha_{32} & 1 - \alpha_{23}
\end{pmatrix}
\]

With \(\alpha\) replacing \(I\), the Cournot equilibrium becomes
\[
\begin{align*}
x^*_1(\alpha) &= \frac{1}{4 - \alpha_{23} - \alpha_{32}}(a - c), \\
x^*_2(\alpha) &= \frac{1 - \alpha_{23}}{4 - \alpha_{23} - \alpha_{32}}(a - c), \\
x^*_3(\alpha) &= \frac{1 - \alpha_{32}}{4 - \alpha_{23} - \alpha_{32}}(a - c).
\end{align*}
\]

Observe that due to the presence of cross-holdings, firm 2 and firm 3’s outputs decrease, so they compete less aggressively than before, but firm 1’s increases. Observe also that total output decreases. However,
\[
\pi^*_2(\alpha) + \pi^*_3(\alpha) = \frac{(2 - \alpha_{23} - \alpha_{32})(a - c)^2}{(4 - \alpha_{23} - \alpha_{32})^2} < \frac{(a - c)^2}{8} = \pi^*_2(I) + \pi^*_3(I).
\]

It follows that the presence of the cross-holdings between firms 2 and 3 make them jointly worse off but leaves firm 1 the only beneficiary.

Let \(0 < \bar{\delta} < 1/2\) denote the maximum amount of equity each firm is allowed to hold in each rival firm. We say that a cross-holding structure \(\alpha\) is permissible if \(0 \leq \alpha_{ij} \leq \bar{\delta}\) for all \(i \neq j\). For reasons as mentioned earlier, we assume that \(\bar{\delta}\) guarantees that \(x^*_i(\alpha) > 0\) for all \(i\) and for all permissible cross-holding structures \(\alpha\).

We consider a notion of pairwise stability for cross-holding structures that guarantees that no two firms have incentives to make bilateral changes.

**Definition 2** We say that a cross-holding structure \(\alpha\) is pairwise stable if it is permissible and for any \(i \neq j\), \((\alpha_{ii}, \alpha_{ij}, \alpha_{ji}, \alpha_{jj})\) solves

\[
\max_{\alpha'} (\pi^*_i(\alpha') + \pi^*_j(\alpha'))
\]

subject to:
\[
\begin{align*}
\alpha'_{ij} + \alpha'_{jj} &= \alpha_{ij} + \alpha_{jj}, \quad 0 \leq \alpha'_{ij} \leq \bar{\delta}, \\
\alpha'_{ii} + \alpha'_{ji} &= \alpha_{ii} + \alpha_{ji}, \quad 0 \leq \alpha'_{ji} \leq \bar{\delta}, \\
\alpha'_{kl} &= \alpha^*_{kl}, \quad kl \neq ii, ij, ji, jj.
\end{align*}
\]

Definition 2 is adapted from Jackson and Wolinsky (1996). In their paper, when a
link between $i$ and $j$ does not exist (which in our setting means that either firm $i$ does not hold any passive ownership in firm $j$ or firm $j$ does not hold any passive ownership in firm $i$), pairwise stability also requires that, taking the existing collection of links as given, $j$ (resp. $i$) be necessarily worse off with the presence of a link between $i$ and $j$ not already presented in the existing collection should $i$ (resp. $j$) become better off. This requirement is automatic in our setting because we require that the link maximize joint profit.

3 Main Results

We assume $P(X) = a - X$ with $0 \leq c < a$ throughout the rest of the paper, in order to focus on firms’ incentives for forming cross-holdings that is driven by the desire to increase market power without any basis for synergy. With symmetric constant marginal cost, it does not lose any generality to have the slope of the demand equal to -1, because we can always normalize the unit for measuring the outputs to make the slope equal to -1 without affecting the analysis.

Given a permissible cross-holding structure $\alpha$, the Cournot equilibrium, $x^*(\alpha)$, satisfies the following first-order conditions

$$P'(X)X_i + \alpha_{ii}[P(X) - c] = 0. \quad (8)$$

Equation (8) together with market demand implies that for all $i$,

$$X^*_i(\alpha) = \sum_{j=1}^{n} \alpha_{ij}x^*_j(\alpha) = \frac{(a - c)\alpha_{ii}}{1 + \sum_{j=1}^{n} \alpha_{jj}}, \quad (9)$$

and

$$P(x^*(\alpha)) = \frac{a + c \sum_{j=1}^{n} \alpha_{jj}}{1 + \sum_{j=1}^{n} \alpha_{jj}}. \quad (10)$$

Thus, by (2), (4), (9), and (10),

$$\pi^*_h(\alpha) = [P(x^*(\alpha)) - c] X^*_i(\alpha) = \frac{\alpha_{ii}(a - c)^2}{(1 + \sum_{j=1}^{n} \alpha_{jj})^2}. \quad (11)$$

By (11), profits from production of the firms are determined by the equity-based share $(X^*_1(\alpha), \cdots, X^*_n(\alpha))$ of total equilibrium output $X^*(\alpha)$. By (9), $X^*_i(\alpha)$ only depends
on $\alpha_{hh}$ for $h = 1, 2, \cdots, n$. Furthermore, $\partial X_i^*(\alpha)/\partial \alpha_{ii} > 0$, $\partial(X_i^*(\alpha) + X_j^*(\alpha))/\partial(\alpha_{ii} + \alpha_{jj}) > 0$, and $\partial X_j^*(\alpha)/\partial(\alpha_{ii} + \alpha_{jj}) > 0$.

**Theorem 1** For all $i$, 

$$\min_{\alpha: \alpha_{ij} = \alpha_{ji} = 0, j \neq i} \pi_i^*(\alpha) = \pi_i^*(I).$$

**Proof.** See Appendix.  

Theorem 1 shows that to be individually rational, total net payment and profit from production for each firm must result in a sum no less than what firm gets in the Cournot equilibrium with no cross-holdings. That is, individual rationality implies that the bilateral formation of pairwise stable cross-holdings is (weakly) Pareto improving as compared with Cournot equilibrium with no cross-holdings.

By (11), a change in cross-holdings between two firms generates two effects on their total profits from production. The output effect is given by

$$[P(X_i^*(\alpha) - c)] \frac{\partial X_i^*(\alpha) + X_j^*(\alpha)}{\partial(\alpha_{ii} + \alpha_{jj})} = \frac{1 + \sum_{k \neq i,j} \alpha_{kk}}{(1 + \sum_k \alpha_{kk})^3} (a - c)^2$$

and the price effect is given by

$$(X_i^*(\alpha) + X_j^*(\alpha)) \frac{\partial P(X_i^*(\alpha))}{\partial(\alpha_{ii} + \alpha_{jj})} = -\frac{\alpha_{ii} + \alpha_{jj}}{(1 + \sum_k \alpha_{kk})^3} (a - c)^2.$$  

Since $\partial (X_i^*(\alpha) + X_j^*(\alpha))/\partial(\alpha_{ii} + \alpha_{jj}) > 0$, and since an increase in $\alpha_{ij} + \alpha_{ji}$ decreases $\alpha_{ii} + \alpha_{jj}$, it follows from (12) and (13) that the output effect of an increase in $\alpha_{ij} + \alpha_{ji}$ is negative while the price effect is positive. To be pairwise stable, it is necessary for the two effects to be balanced whenever constraints on cross-holdings between firms $i$ and $j$ are not binding. Indeed, combining (11) with (12) and (13) yields

$$\frac{\partial (\pi_i^*(\alpha) + \pi_j^*(\alpha))}{\partial(\alpha_{ii} + \alpha_{jj})} = \frac{1 + \sum_{h \neq i,j} \alpha_{hh} - (\alpha_{ii} + \alpha_{jj})}{(1 + \sum_h \alpha_{hh})^3} (a - c)^2.$$  

Since $\alpha_{hh} \geq 1 - \bar{\delta} > 1/2$ for all $h$ and since $\sum_{h \neq i,j} \alpha_{hh} = 0$ if $n = 2$, the right-hand-side of (14) is negative when there are two firms. Thus, with $n = 2$, total profit increases as $\alpha_{11} + \alpha_{22}$ decreases. This implies that $\alpha_{12} = \alpha_{21} = \bar{\delta}$ for $\alpha$ to be pairwise stable because $\alpha_{ij}$ increases as $\alpha_{jj}$ decreases. On the other hand, $1 + \sum_{h \neq i,j} \alpha_{hh} > 2$ if $n \geq 4$ and $\alpha$ is permissible. Hence, the right-hand-side of (14) is positive when there are four or more firms. Consequently, pairwise stability of $\alpha$ implies $\alpha_{ii} = 1$ for all $i$. This establishes:
Theorem 2 A cross-holding structure $\alpha$ is pairwise stable if and only if $\alpha_{12} = \alpha_{21} = \bar{\delta}$ when $n = 2$ and $\alpha_{ii} = 1$ for all $i$ when $n \geq 4$.

When there are three firms, the output and price effects can be balanced at some cross-holding structures. However, additional conditions are needed for these cross-holding structures to be pairwise stable. It turns out that there is a wide range of pairwise stable cross-holding structures with three firms.

Theorem 3 Assume $n = 3$. Then, $\alpha$ is pairwise stable if and only if $\alpha_{ii} = 1$, $\alpha_{ij} = \alpha_{ik} \leq \bar{\delta}$, and $\alpha_{jk} = \alpha_{kj} = 0$.

Proof. See Appendix.

Theorem 3 shows, in particular, that $I$ is pairwise stable which results in the usual triopoly Cournot equilibrium with no cross-holdings. On the other hand, $\alpha$ in (6) with $\delta \leq \bar{\delta}$ is also pairwise stable. As illustrated before, the Cournot equilibrium given $\alpha$ satisfies

$$x_1^*(\alpha) = \frac{(1 - 2\delta)(a - c)}{2(2 - \delta)}.$$

Accordingly, $x_1^*(\alpha) = 0$ when $\delta = 1/2$. Hence, when $\bar{\delta}$ is close to $1/2$, the Cournot equilibrium with $\delta = \bar{\delta}$ is close to be the usual duopoly Cournot equilibrium having firms 2 and 3 as the duopolistic firms. The major part of the proof of Theorem 3 is for establishing the star shape of pairwise stable cross-holding structures as summarized in this theorem.

4 Conclusion

In this paper we analyzed pairwise stable and individually rational cross-holdings in Cournot oligopoly. Individual rationality in this paper means that no firm is made worse off from participating in cross-holdings than not participating. Pairwise stability means that no two firms can increase total profits by a bilateral change in passive ownership in each other. The pairwise stability depends on two opposite effects on total profits generated from a change in cross-holdings between two firms. They are the output effect and the price effect. The price effect increases total profits while the output effect decreases total profits.

We showed that the individual rationality of a cross-holding structure implies that the total profit of each firm is no less than what it gets in the Cournot equilibrium with
no cross-holdings. It follows that the formation of an individually rational cross-holding structure is (weakly) Pareto improving as compared with the Cournot equilibrium with no cross-holdings. We also showed that with linear demand and cost functions and with Cournot competition on output market, the ability to bilaterally form cross-holdings can lead to collusive outcomes when there are no more than three firms, but the outcome is as competitive as the usual Cournot equilibrium with no cross-holdings when there are four or more firms. That is, Cournot oligopoly becomes more collusive due to bilateral formation of cross-holdings when the industry is more concentrated.

Appendix: Proofs

**Proof of Theorem 1:** By (11), given any collection \( \alpha_i, \alpha_{ii} = 1 \) implies

\[
\pi^*_i(\alpha) = \frac{(a - c)^2}{(1 + \sum_{k=1}^n \alpha_{kk})^2}.
\]

It follows that firm \( i \)'s profit is decreasing in other firms’ retained equity shares. Hence, the minimum profit of firm \( i \) is achieved at \( \alpha_{jj} = 1 \) for all \( j \neq i \). This together with \( \alpha_{ii} = 1 \) implies that the minimum profit from retaining all equity is firm \( i \)'s Cournot equilibrium profit without any cross-holding. \( \blacksquare \)

To prove Theorem 3, we need the following three lemmas.

**Lemma 1** A cross-holding structure \( \alpha \) is pairwise stable if and only if for any \( i \neq j \), \( (\alpha_{ij}, \alpha_{jj}) \) solves

\[
\begin{align*}
\max \pi^*_i(\alpha') + & \pi^*_j(\alpha') \\
subject to: & \alpha'_{ij} + \alpha'_{jj} = \alpha_{ij} + \alpha_{jj}, 0 \leq \alpha'_{jj} \leq \tilde{\delta}, \\
& \alpha'_{kl} = \alpha_{kl}, kl \neq ij, jj.
\end{align*}
\]

\((A1)\)

**Proof.** By (11), for any \( i \neq j \),

\[
\pi^*_i(\alpha) + \pi^*_j(\alpha) = \frac{(\alpha_{ii} + \alpha_{jj})(a - c)^2}{(1 + \sum_{k=1}^n \alpha_{kk})^2}.
\]

It follows that the total profit of firms \( i \) and \( j \) is a concave function of the sum of their retained shares. Consequently, if bilateral cross-holding changes satisfying (7) can increase the total profit of firms \( i \) and \( j \), then so can bilateral cross-holding changes satisfying constraints in \((A1)\). The converse is clearly also true. \( \blacksquare \)
Lemma 2 A cross-holding structure $\alpha$ is pairwise stable if and only if for all $i \neq j$ and for all bilateral changes $(d\alpha_{ij}, d\alpha_{jj})$ such that $d\alpha_{ij} + d\alpha_{jj} = 0$ and $\alpha_{ij} + d\alpha_{ij} \leq \bar{\delta}$,

$$1 + \sum_{k \neq i,j}^{n} \alpha_{kk} - (\alpha_{ii} + \alpha_{jj}) \leq 0.$$  \hfill (A2)

Proof. By (7), $\pi^*_i(\alpha) + \pi^*_j(\alpha)$ is concave and differentiable in $\alpha_{jj}$. It follows that given $\alpha$, for $(\alpha_{ij}, \alpha_{jj})$ to solve (A1), it is necessary and sufficient

$$d[\pi^*_i(\alpha) + \pi^*_j(\alpha)] \leq 0$$

which in turn is equivalent to

$$1 + \sum_{k \neq i,j}^{n} \alpha_{kk} - (\alpha_{ii} + \alpha_{jj}) \leq 0$$

Lemma 3 Assume $n \geq 3$. If $\alpha$ is permissible and pairwise stable, then $\alpha_{ij} = 0$ for all $j \neq i$ whenever $\alpha_{hi} \neq 0$ for some $h \neq i$.

Proof. Let $\alpha$ be pairwise stable. We prove the lemma in two steps.

Step 1: $\alpha_{hi} \alpha_{ij} = 0$ for any three different firms $h$, $i$, and $j$.

Suppose on the contrary there exist three distinct firms $h$, $i$, and $j$ for which $\alpha_{hi} \alpha_{ij} \neq 0$. Then, $\alpha_{hi} > 0$ implies that $d\alpha_{ii} > 0$ can be feasibly arranged between firm $h$ and firm $i$. Thus, by (A2),

$$1 + \sum_{k \neq h,i} \alpha_{kk} \leq \alpha_{hh} + \alpha_{ii}.$$  \hfill (A3)

Similarly, since $\alpha_{ij} > 0$, (A2) implies

$$1 + \sum_{k \neq i,j} \alpha_{kk} \leq \alpha_{ii} + \alpha_{jj}.$$  \hfill (A4)

Putting (A3) and (A4) together, we have $\alpha_{ii} \geq 1$ which contradicts $\alpha_{hi} > 0$.

Step 2: $\alpha_{ij} \alpha_{ji} = 0$. 

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Suppose on the contrary there exists two distinct firms $i$ and $j$ such that $\alpha_{ij}\alpha_{ji} \neq 0$. Then, $\alpha_{ij} > 0$ and so, $d\alpha_{jj} > 0$ can be feasibly arranged between firms $i$ and $j$. It follows from (A2)

$$1 + \sum_{h \neq i,j} \alpha_{hh} \leq \alpha_{ii} + \alpha_{jj}. \tag{A5}$$

Since $\alpha_{ij}\alpha_{ji} \neq 0$ implies $\alpha_{ii} < 1$ and $\alpha_{jj} < 1$, it follows from (A5) that $\alpha_{hh} < 1$ for all firms $h \neq i, j$. Fix firm $h \neq i, j$. Then, there exists a firm $k \neq h$ such that $\alpha_{kh} > 0$. Since $\alpha_{ij}\alpha_{ji} \neq 0$, it follows from Step 1 and $\alpha_{kh} > 0$ that $k \neq i, j$, which is impossible when $n = 3$. Suppose $n > 3$. Then, $\alpha_{kk} < 1$ implies that there exists a firm $l \neq i, j$ such that $\alpha_{lk} > 0$. Since $\alpha_{kh} > 0$, it follows from Step 1 that $m = k$. Thus, by (A2),

$$1 + \sum_{h \neq k,l} \alpha_{hh} \leq \alpha_{kk} + \alpha_{ll}. \tag{A6}$$

Putting (A5) and (A6) together, we have $2 \leq 0$ which yields the desired contradiction. 

**Proof of Theorem 3**: The sufficiency of the conditions in Theorem 3 follows easily from (A2). Let $\alpha$ be given. If $\alpha_{hh} < 1$ for all $h$. Then, each firm’s equity shares are cross held by another firm. However, in that case, Lemma 2 implies that no firm cross holds any other firm’s equity shares. This is clearly a contradiction. Now suppose $\alpha_{ii} = 1$. If $\alpha_{jj} = \alpha_{kk} = 1$, then the conditions in the theorem are satisfied. Without loss of generality, assume $\alpha_{ij} > 0$. Then, $d\alpha_{jj} > 0$ is permissible for firms $i$ and $j$. Consequently, by (A2),

$$1 + \alpha_{kk} \leq \alpha_{ii} + \alpha_{jj} \iff \alpha_{kk} \leq \alpha_{jj}. \tag{A7}$$

which implies $\alpha_{kk} < 1$. Since $\alpha_{ij} > 0$, by Lemma 2, $\alpha_{jk} = 0$. We thus conclude $\alpha_{ik} > 0$. Consequently, $d\alpha_{kk} > 0$ is permissible for firms $i$ and $k$. By (A2),

$$1 + \alpha_{jj} \leq \alpha_{ii} + \alpha_{kk} \iff \alpha_{jj} \leq \alpha_{kk}. \tag{A8}$$

Putting (A7) and (A8) together, we have $\alpha_{jj} = \alpha_{kk}$ which in turn implies $\alpha_{ij} = \alpha_{ik}$. 

13
References


