A Extended Proofs of Main Theoretical Results

1 Monopoly: Some Known Results

Let the gross value of investment V be governed by the stochastic differential equation

$$dV = \alpha V dt + \sigma V dz, \tag{A1}$$

where z is the standard Wiener process. A key insight from real options theory[1] is that when future values are discounted at rate $\rho \geq 0$, the expected present value of waiting for appreciation $R > 1$ is $R^{-\beta}$, where

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left[\frac{\alpha}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}} \geq 0. \tag{A2}$$

Thus β is a composite discount parameter that incorporates the stochastic appreciation trend α and volatility σ as well as impatience ρ. In particular, if the current gross value is V_o and the investor will seize the opportunity when V hits threshold $V_1 > V_o$, then the discount factor is

$$\left[\frac{V_o}{V_1}\right]^\beta. \tag{A3}$$

When there are no rivals ($n = 0$), an investor with avoidable fixed cost $C > 0$ seeks to maximize the expected discounted profit $E\left[(V - C)\, e^{-\rho t}\right]$. In view of (A3), the problem reduces to finding a threshold V_1 to maximize $[V_1 - C][V_o/V_1]^\beta$. The associated first order condition is

$$0 = \frac{1}{V_1 - C} - \frac{\beta}{V_1}, \tag{A4}$$

with solution

$$V_1 = V_M \equiv \left(\frac{\beta}{\beta - 1}\right) C. \tag{A5}$$

If $\beta > 1$, then (A5) gives the optimal threshold we seek. If $\beta \in [0, 1]$, e.g., if $\alpha \geq \rho$ in (A2), then the expected discounted profit increases in V_1 over the entire domain $V_1 \geq C$, so there is no finite optimal threshold.

[1] See Samuel Karlin and Howard M. Taylor (1975, 357-364) for a derivation using the Laplace transform and the martingale theorem; see also Appendix A of our working paper, Steven T. Anderson, Daniel Friedman and Ryan Oprea (2009), for a streamlined two-page derivation of the monopoly solution.

Note that (A4-A5) can also usefully be written

$$V_M(C) - \beta[V_M(C) - C] = 0. \tag{A6}$$

2 The Preemption Game: Some Useful Constructs

Let $\Gamma[\beta, n, H]$ be the preemption game described informally in section I.B of the text. A bit more formally, in $\Gamma[\beta, n, H]$ each player ("investor") $i = 1, ..., n+1$ chooses a measurable threshold function $V_i : [C_L, C_U] \to [0, \infty)$, draws realized cost C_i independently from cumulative distribution H on $[C_L, C_U]$, and seizes the investment opportunity at the first time t_i that V in (A1) hits $V_i(C_i)$. In view of (A3), player i's (expected) payoff is

$$[V_i(C_i) - C_i]\left[\frac{V_o}{V_i(C_i)}\right]^{\beta} \Pr[t_i < t_j \quad \forall j \neq i], \tag{A7}$$

given initial value V_o in (A1).

We seek a symmetric Bayesian Nash equilibrium (SBNE) of $\Gamma[\beta, n, H]$, that is, a single threshold function V^* such that if all other investors $j \neq i$ choose $V_j = V^*(C_j)$ then investor i maximizes payoff (A7) at any cost draw $C_i \in [C_L, C_U]$ by choosing threshold $V^*(C_i)$. Theorem 1 asserts that such a SBNE V^* exists and is unique, and Theorem 2 establishes some of its general properties. The remainder of this section derives some constructs—in particular, a boundary value problem and a recursion formula—that will help prove those theorems and will provide additional insight.

To begin, suppose all rivals use the same increasing differentiable threshold function $\tilde{V}(C_j)$. At some arbitrary time t_o, let $V_o = V(t_o)$ be the current gross investment value, let $\hat{V} = \max_{s \in [0, t_o]} V(s)$ be the highest value yet observed, and let $\hat{C} = \tilde{V}^{-1}(\hat{V})$ be highest cost draw that would already have led a rival to invest. Investor i can assume that all $C_j > \hat{C}$, since otherwise the game is already over and his choice is moot.

The unconditional probability that any one rival j has cost C_j higher than C is $1 - H(C)$, and the probability conditional on $C_j > \hat{C}$ is $\frac{1-H(C)}{1-H(\hat{C})}$. If investor i chooses threshold $V_1 = \tilde{V}(m)$, the probability that any particular rival will not preempt therefore is $\frac{1-H(m)}{1-H(\hat{C})}$. Thus the conditional probability that none of the n rivals will preempt at that threshold is

$$\Pr[t_i < t_j \quad \forall j \neq i] = \left[\frac{1 - H(m)}{1 - H(\hat{C})}\right]^n \tag{A8}$$

When impatience is due only to expiration hazard, the expression (A3) is simply the probability that the investment opportunity does not expire (or that "Nature does not preempt") before the threshold V_1 is hit. In this case, at current state $[V_o, \hat{V} = \tilde{V}(\hat{C})]$, the probability that investor i "wins" the preemption game is

$$1 - G(m) \equiv \left[\frac{V_o}{\tilde{V}(m)}\right]^{\beta} \left[\frac{1 - H(m)}{1 - H(\hat{C})}\right]^{n}. \tag{A9}$$

More generally, (A3) discounts the investor's future values for everything other than preemption by other players, which is captured in (A8), so the product (A9) is the overall discount factor.

Assume that the cost distribution H has density h with full support on the interval $[C_L, C_U]$, where $0 < C_L < C_U < \infty$. The definition (A9) of G then ensures that G has a positive and continuous derivative g. Note also that $1 - G(C_U) = 0$ since $1 - H(C_U) = 0$.

Given investment cost C_i, investor i's problem reduces to finding a threshold $V_1 = \tilde{V}(m)$ that solves

$$\max_{m \in [\hat{C}, C_U]} \left[\tilde{V}(m) - C_i\right] [1 - G(m)]. \tag{A10}$$

The FOC is

$$\tilde{V}'(m)[1 - G(m)] - \left[\tilde{V}(m) - C_i\right] g(m) = 0. \tag{A11}$$

In SBNE, investor i will find it advantageous to use the same threshold function $\tilde{V}$ as the other investors. Accordingly, insert the "truthtelling" condition $m = C_i$ into (A11), and simplify notation by setting $C_i = C$, to obtain

$$\tilde{V}'(C)[1 - G(C)] - \left[\tilde{V}(C) - C\right] g(C) = 0. \tag{A12}$$

Rearranging slightly, write

$$-Cg(C) = \tilde{V}'(C)[1 - G(C)] - \tilde{V}(C)g(C) = \frac{d}{dC}[\tilde{V}(C)(1 - G(C))], \tag{A13}$$

and integrate both sides of (A13) from $C = C_i$ to C_U, using $1 - G(C_U) = 0$, to obtain

$$-\int_{C}^{C_U} yg(y)dy = 0 - \tilde{V}(C)[1 - G(C)]. \tag{A14}$$

Hence the SBNE threshold function $\tilde{V} = V^*$ must satisfy

$$V^*(C) = \frac{1}{1-G(C)} \int_C^{C_U} yg(y)dy \equiv E_G[y|y > C]. \tag{A15}$$

Since H and hence G are smooth strictly increasing functions, equation (A15) ensures that V^* is also smooth and increasing in C.

Of course, the function G is itself defined in terms of V^*.[2] To obtain an explicit recursion formula, first integrate by parts to get $\int_C^{C_U} yg(y)dy = C_U - CG(C) - \int_C^{C_U} G(y)dy$. Then (A15) reads

$$\begin{aligned} V^*(C) &= \frac{1}{1-G(C)} \left[C_U - CG(C) - \int_C^{C_U} 1 - \left(\frac{V_o}{V^*(y)}\right)^\beta \left(\frac{1-H(y)}{1-H(\hat{C})}\right)^n dy \right] \\ &= \frac{1}{1-G(C)} \left[C[1-G(C)] + \int_C^{C_U} \left(\frac{V_o}{V^*(y)}\right)^\beta \left(\frac{1-H(y)}{1-H(\hat{C})}\right)^n dy \right] \\ &= C + \int_C^{C_U} \left[\frac{V^*(C)}{V^*(y)}\right]^\beta \left[\frac{1-H(y)}{1-H(C)}\right]^n dy. \end{aligned} \tag{A16}$$

The last expression shows that $V^*(C)$ is equal to C plus a positive markup, which shrinks to 0 as C approaches its upper endpoint $C = C_U$.

Remark A1. Note that V_o and $\hat{V} = V^*(\hat{C})$ drop out of the formula (A16) for V^*. This reflects the fact that the opportunity to observe rivals' actions does not influence thresholds; $V^*(C)$ can be set as soon as the investor draws cost C, and subsequent observations are irrelevant. The logic parallels the strategic isomorphism of Dutch and first price auctions.

To obtain the boundary value problem satisfied by V^*, first insert (A9) into (A10) to write the objective function as in the text:

$$F\left(m|C_i, n\right) = \left[V^*\left(m\right) - C_i\right] \left[\frac{V}{V^*\left(m\right)}\right]^\beta \left[\frac{1-H\left(m\right)}{1-H\left(\hat{C}\right)}\right]^n. \tag{A17}$$

Use the product rule to take the derivative of Equation (A17) with respect to m and cancel like terms (or, alternatively, take the derivative of $\ln F$) and evaluate at the "truth-telling" point $m = C_i$, to obtain the FOC:

$$\frac{V^{*\prime}\left(C_i\right)}{\left[V^*\left(C_i\right) - C_i\right]} - \frac{\beta V^{*\prime}\left(C_i\right)}{V^*\left(C_i\right)} - \frac{nh\left(C_i\right)}{\left[1-H\left(C_i\right)\right]} = 0. \tag{A18}$$

[2] Here we depart from the auction literature, e.g., Vijay Krishna (2002, 14-19).

Solve (A18) for $V^{*\prime}$ to obtain Equation (5), reproduced here for convenience:

$$V^{*\prime}(C_i) = \frac{[V^*(C_i) - C_i]\, V^*(C_i)}{V^*(C_i) - \beta\,[V^*(C_i) - C_i]} \times \frac{nh(C_i)}{[1 - H(C_i)]} \tag{A19}$$

As noted in the text, the boundary condition

$$V^*(C_U) = C_U \tag{A20}$$

comes from the economics of the situation. At the highest possible cost realization, the existence of rivals known to have equal (or lower) cost induces Bertrand competition and drives the markup to zero.

Remark A2. To obtain V^* numerically, one can use the Euler method of integrating the ODE (A19) backward from the upper boundary value (A20). Alternatively, one can take an initial approximation such as the auction solution $\bar{V}$, substitute it for V^* in the last expression in (A16) to obtain a better approximation, and iterate. The BNE threshold function V^* is a fixed point of this mapping.

3 Proof of Theorem 1

Theorem 1. *Let the cumulative distribution function H have a continuous density h with full support $[C_L, C_U]$, where $0 < C_L < C_U < \infty$. Let it be common knowledge among all investors $i = 1, ..., n+1$ that i's investment cost C_i is an independent random variable with distribution H that is observed only by investor i. Then*

1. *for any $\beta \geq 0$, boundary value problem (A19-A20) has a unique solution $V^* : [C_L, C_U] \rightarrow R$,*
2. *a function V^* satisfies the recursion equation (A16) iff it solves the boundary problem (A19-A20), and*
3. *the premption game $\Gamma[\beta, n, H]$ has a symmetric Bayesian-Nash equilibrium in which each investor i's threshold is V^* evaluated at realized cost C_i.*

Lemma A1. *Assume that $\beta > 1$, that all rivals use a threshold function with inverse γ such that $\gamma' > 0$, and that the hypotheses of Theorem 1 hold. Let the threshold value y maximize the competitor's payoff given cost realization $C \in [C_L, C_U]$. Then $y < V^*_M(C)$.*

Proof of Lemma. Write (A4) as

$$0 = \frac{1}{x - C} - \frac{\beta}{x}. \tag{A21}$$

and write (A17) as

$$[x - C]\left[\frac{V}{x}\right]^{\beta}\left[\frac{1 - H(\gamma(x))}{1 - H(\hat{C})}\right]^{n}. \tag{A22}$$

Since the threshold value $x = y$ maximizes (A22), it must satisfy the FOC

$$0 = \frac{1}{x - C} - \frac{\beta}{x} - \frac{nh[\gamma(x)]\gamma'(x)}{1 - H[\gamma(x)]}. \tag{A23}$$

By (A21), at $x = V_M^*$ the RHS of (A23) reduces to

$$-\frac{nh[\gamma(x)]\gamma'(x)}{1 - H[\gamma(x)]} < 0. \tag{A24}$$

Since the RHS of (A21) is negative for $x > V_M$, no value of $x \geq V_M$ can satisfy the FOC (A23). On the other hand, since the first RHS term in (A23) goes to $+\infty$ as $x \searrow C$ while the other terms remain bounded, the continuity of the RHS in x guarantees a solution to (A23) at some value $x = y \in (C, V_M)$. $\diamondsuit$

Proof of Theorem 1. The key step in part 1 is to show that the RHS of the ODE (A19) is Lipschitz continuous in V^*. A first potential problem is that the denominator factor $1 - H(C) \searrow 0$ as $C \nearrow C_U$. But given the boundary condition (A20), the numerator factor $V^*(C) - C \searrow 0$ also. Indeed, we have

$$\begin{aligned} V^{*\prime}(C_U) &= \lim_{C \nearrow C_U} \frac{[V^*(C) - C]V^*(C)}{V^*(C) - \beta[V^*(C) - C]} \times \frac{nh(C)}{1 - H(C)} \\ &= \frac{nh(C_U)V^*(C_U)}{V^*(C_U) - \beta[V^*(C_U) - C_U]} \lim_{C \nearrow C_U} \frac{V^*(C) - C}{1 - H(C)} \\ &= \frac{nh(C_U)}{1 - \beta[1 - \frac{C_U}{C_U}]} \frac{V^{*\prime}(C_U) - 1}{[-h(C_U)]} = -n[V^{*\prime}(C_U) - 1], \end{aligned} \tag{A25}$$

using L'Hospital's rule. Hence $V^{*\prime}(C_U) = n/(n + 1)$, (independent of β and H!). In particular, we have Lipschitz continuity in an ϵ neighborhood of the upper boundary point.

A second potential problem is that the other denominator factor $V^*(C) - \beta[V^*(C) - C]$ might be zero (or negative). For $\beta \leq 1$ the factor obviously is positive. Lemma A1 assures us that the expression is also positive when $\beta > 1$:

$$\begin{aligned} V^*(C) - \beta[V^*(C) - C] &= y - \beta[y - C] \\ = \beta C - (\beta - 1)y &> \beta C - (\beta - 1)V_M(C) \\ &= V_M(C) - \beta[V_M(C) - C] = 0, \end{aligned} \tag{A26}$$

since $V^*(C) = y < V_M(C)$ by the Lemma and the last expression is 0 by (A6).

Since the denominator is continuous on the closed interval $[C_L, C_U - \epsilon]$, it achieves a positive minimum value and thus is bounded away from zero. It is now clear that the RHS of the ODE (A19) is positive, bounded and Lipschitz continuous in V^*. The classic Picard-Lindelof theorem (e.g., see Chapter 8 of Morris W. Hirsch and Stephen Smale, 1974) then guarantees that a solution V^* to the boundary problem (A19-A20) exists and is unique.

For the second part of the Theorem, suppose that V^* satisfies the recursion equation (A16). Note that the integrand $f(C, y)$ in the last term of (A16) is the product of two increasing functions of C, so $\frac{\partial f(C,y)}{\partial C} > 0$. Indeed, using the product rule, one obtains $\frac{\partial f(C,y)}{\partial C} = \left(\frac{\beta V'(C)}{V(C)} + \frac{nh(C)}{1-H(C)}\right) f(C, y)$. Differentiating both sides of (A16), we get

$$\begin{aligned} V^{*\prime}(C) &= 1 - f(C, C) + \int_C^{C_U} \frac{\partial f(C, y)}{\partial C} dy = \int_C^{C_U} \frac{\partial f(C, y)}{\partial C} dy \\ &= \left(\frac{\beta V^{*\prime}(C)}{V^*(C)} + \frac{nh(C)}{1 - H(C)}\right) \int_C^{C_U} f(C, y) dy \\ &= \left(\frac{\beta V^{*\prime}(C)}{V^*(C)} + \frac{nh(C)}{1 - H(C)}\right) [V^*(C) - C] > 0 \end{aligned} \tag{A27}$$

for all $C \in [C_L, C_U)$. The last equality follows from subtracting C from both sides of (A16). Divide (A27) through by the markup $V^*(C) - C > 0$ to obtain (A18). As noted earlier, this can be rewritten as the ODE (A19). In the limit as $C \nearrow C_U$, the integral term vanishes in (A16) and we obtain the boundary condition (A19).

Conversely, suppose that V^* solves the boundary value problem (A19-A20). Then the derivation (A12-A16) ensures that it also satisfies the recursion formula (A16).

To establish part 3 of the Theorem, let $W(C, m)$ be the expected payoff to investor i when she draws cost C and and employs threshold $V_1 = V^*(m)$, assuming each other investor j sets threshold $V^*(C_j)$. We complete the proof by showing that setting $m = C$, i.e., truthtelling, always maximizes $W(C, m)$.

Assume that $m > \hat{C}$; otherwise the investor already would have ended the game. Consequently $V^*(m) > \hat{V} \geq V_o$ and, by (A10) and (A15),

$$\begin{aligned} W(C,m) &= [V^*(m) - C][1 - G(m)] \\ &= \int_m^{C_U} yg(y)dy - C + CG(m) \\ &= [C_U - C] + [C - m]G(m) - \int_m^{C_U} G(y)dy, \end{aligned} \tag{A28}$$

where the last expression uses integration by parts: $\int_m^{C_U} yg(y)dy = C_U - mG(m) - \int_m^{C_U} G(y)dy$. But (A28) and the Mean Value Theorem (MVT) yield a point x such that

$$\begin{aligned} W(C,C) - W(C,m) &= [m - C]G(m) + \int_m^{C} G(y)dy \\ &= [m - C]G(m) + [C - m]G(x) \\ &= [m - C][G(m) - G(x)] \geq 0, \end{aligned} \tag{A29}$$

since G is increasing and the number x guaranteed by the MVT is between m and C. Hence $m = C$ is indeed a best response and V^* is indeed a symmetric BNE. ♢

Remark A3. Theorem 1 imposes the hypothesis that H have a continuous positive density on the entire support interval, and that the upper endpoint is finite and the lower endpoint is positive. It follows from Lusins theorem (e.g., Walter Rudin, 1966, 53-54) that such functions are a dense subset of all distribution functions on $[0, \infty)$, so this restriction is not especially onerous. Still, it would be of interest to explicitly consider distributions with discrete support (e.g., a finite set of cost "types") or asymmetric cases in which some investors are known to have different cost distributions than other investors. Perhaps "ironing" techniques in the spirit of Roger B. Myerson (1981) would be useful for such extensions.

4 Proof of Theorem 2

Proof of Theorem 2. The RHS of ODE (A19) is positive and continuous, so its solution V^* is increasing and continuously differentiable. The Marshallian lower bound is also obvious: a threshold below cost implies negative profit, but zero profit can be assured in this game by never investing. Hence any BNE threshold must be at least the realized cost.

To see that V_M is an upper bound for the SBNE threshold, note that the investor seeks threshold

V_1 to maximize

$$[V_1 - C]\left[\frac{V_o}{V_1}\right]^\beta \left[\frac{1 - H(V^{*-1}(V_1))}{1 - H(\hat{C})}\right]^n, \tag{A30}$$

The optimum threshold value $V_1 = V^*(C)$ must satisfy the FOC

$$0 = \frac{1}{V_1 - C} - \frac{\beta}{V_1} - \frac{nh[V^{*-1}(V_1)]V^{*-1\prime}(V_1)}{1 - H[V^{*-1}(V_1)]}. \tag{A31}$$

By (A4), at $V_1 = V_M$ the RHS of (A31) reduces to

$$-\frac{nh[V^{*-1}(V_1)]V^{*-1\prime}(V_1)}{1 - H[V^{*-1}(V_1)]} < 0. \tag{A32}$$

Since the RHS of (A4) is negative for $V_1 > V_M(C)$, no value of $V_1 \geq V_M$ can satisfy the FOC (A31). On the other hand, since the first RHS term in (A31) goes to $+\infty$ as $V_1 \searrow C$ while the other terms remain bounded, the continuity of the RHS in V_1 guarantees a solution to (A31) at some value $V_1 \in (C, V_M(C))$, i.e., V_M is indeed an upper bound.

A similar argument shows that the auction bid function $\bar{V}$ is also an upper bound. Evaluating the RHS of (A31) at $V_1 = \bar{V}(C)$, the first and third terms disappear due to the FOC that characterizes $\bar{V}$, and the remaining term, $-\beta/V_1$, is negative. Again, higher values of V_1 only make the RHS of (A31) more negative, but we know that the RHS is positive for sufficiently small values of V_1. Hence the intermediate value theorem again guarantees a solution to (A31) at some value $V_1 \in (C, \bar{V}(C))$.

Recall that (A25) showed that $V^{*\prime}(C_U) = n/(n+1)$, independent of β. Since $V^* = \bar{V}$ in the special case $\beta = 0$, we have $\bar{V}'(C_U) = n/(n+1) = V^{*\prime}(C_U)$. Of course, $V^*(C_U) = C_U = \bar{V}(C_U)$, so $\bar{V}$ and V^* are indeed tangent at C_U. $\diamondsuit$

Remark A4. It is well known from auction theory that the bid function $\bar{V}(C)$ converges to the Marshallian threshold function $V^0(C) = C$ as the number of rivals $n \to \infty$. A simple consequence of Theorem 2 is that the same is true of the BNE threshold function $V^*(C)$. As n gets large, its upper bound $\bar{V}$ converges to its lower bound V^0, so in the large number limit, $V^*(C) = V^0(C) = C$. That is, markups converge to zero as the number of rivals increases.

Remark A5. The main results from Anderson (2003) are subsumed in this Appendix, with one exception. He obtains a comparative static result for mean-preserving spreads of the cost distribution H.

5 Brownian Parameters

Dixit and Pindyck (1994, pp 69-70) show that the deviation of the uptick probability p from 0.5, times the distance 2η between an uptick and downtick, corresponds to the Brownian drift rate α:

$$\alpha = \lim_{\Delta t \to 0} \frac{(2p-1)\,\eta}{\Delta t}. \tag{A33}$$

The Brownian volatility σ comes mainly from the stepsize η but when p differs from 0.5 we must also account for binomial variance $p\,(1-p)$. The exact expression is

$$\sigma^2 = \lim_{\Delta t \to 0} \frac{4p\,(1-p)\,\eta^2}{\Delta t}. \tag{A34}$$

The relation between expiration probability $q \in [0,1]$ and the discount parameter $\rho > 0$ is obtained as follows. By definition of discount, 1 unit value received a time unit from now is equivalent to $e^{-\rho}$ units received immediately. The usual interpretation of such impatience is the foregone interest payments available in a financial market. For practical reasons, we turn to an alternative interpretation (e.g., David Kreps, 1990, 505-6): the discount rate arises from the possibility that the opportunity will expire. If that event has probability Q per unit time, then the expected value of 1 unit due a time unit in the future is $1-Q$. With $T = 1/\Delta t$ time steps per unit time, and expiration probability q per time step, the discount factor is $e^{-\rho} = 1 - Q = (1-q)^T = (1-q)^{1/\Delta t}$. Solving for ρ we obtain

$$\rho = \frac{-\ln(1-q)}{\Delta t}. \tag{A35}$$

Note that Q is expressed per unit time, while q corresponds to a fixed Δt. Hence in (A35) we don't take the limit $\Delta t \to 0$.

B Supplemental Mathematical Results

1 Concavity

The text following the statement of Theorem 2 listed conditions guaranteeing that the threshold function is concave. We now formalize and prove that result (the only non-trivial part of the Corollary).

Proposition B1. *Suppose that the hypotheses of Theorem 1 hold and that the density h of the* *cumulative distribution function H is non-increasing. Then the BNE threshold function $V^*(C)$ is concave.*

Proof. A sufficient condition for concavity is that $V^{*\prime\prime}(C) \leq 0$, so it sufficies to show that the derivative with respect to C of the RHS of (A19) is not positive. Direct computation reveals that that derivative is

$$\frac{N(C|H,\beta)}{D} = \frac{\{[V^* - \beta\,(V^* - C)]\,[1-H]\} \times \{(nh)\,[(2V^* - C)\,V^{*\prime} - V^*] + [(nh^\prime)\,(V^* - C)\,V^*]\}}{\{V^* - \beta\,[V^* - C] \times [1-H]\}^2}$$
$$- \frac{\{[(1-\beta)V^{*\prime} + \beta]\,[1-H] - (h)\,[V^* - \beta(V^* - C)]\} \times \{(nh)\,(V^* - C)\,V^*\}}{\{V^* - \beta\,[V^* - C] \times [1-H]\}^2}. \quad \text{(B1)}$$

The proof of Theorem 1 shows that $V^* - \beta[V^* - C]$ is bounded below by a positive number, so clearly the denominator D is positive. Thus it suffices to show that the numerator $N \leq 0$.

Now note that $h^\prime$ appears only once in N and that the other factors in that term are positive. Hence if the proposition holds for a uniform distribution, then it holds *a forteriori* for one with decreasing density. The rest of this proof therefore assumes that the distribution is uniform.

Write

$$N(C|H,\beta) = N(C|H,0) + \{[-\beta\,(V^* - C)]\,[1-H]\} \times (nh)\left[(2V^* - C)\,V^{*\prime} - V^*\right]$$
$$- \left\{\left[\beta - \beta V^{*\prime}\right][1-H] + (h)\beta(V^* - C)\right\} \times \{(nh)\,(V^* - C)\,V^*\} \quad \text{(B2)}$$

We know that the (Vickrey) bid function is linear and hence $N(C|H,0) = 0$ when H is uniform. Factoring out $nh\beta(V^* - C)$ from (B2), and simplifying, we obtain

$$N(C|H,\beta) = -nh\beta(V^* - C)^2\left[(1-H)V^{*\prime} + hV^*\right], \quad \text{(B3)}$$

which is strictly negative when $\beta > 0$. $\diamondsuit$

2 Constant Markups

The BNE strategy is not as complex as some sorts of feasible strategies. For example, it is not contingent on time elapsed, nor on the current value of the Brownian motion (as long as it is below the threshold!), nor on the history observed so far. However, the BNE strategy relies on a non-linear function of realized cost. It may be too complex for human investors to discover.

Therefore it may be worth analyzing a restriction of the preemption game to simple 1-dimensional strategy spaces. Here each investor chooses a constant additive markup, i.e., a profit aspiriation $k \geq 0$, and sets the threshold $V^R(C_i, k) = C_i + k$. In all other respects, the game is the same as before.

A symmetric Nash Equilibrium (NE)[3] of the restricted game is a markup k^* that is a best response to itself. To characterize it, suppose that other investors choose markup $k > 0$ and let investor i consider possible deviations $k_i = k + x$ for any real number x. Her objective function is the expected payoff

$$E_H F^R(x|k,n) = (k+x)\int_{C_L}^{C_U}\left[\frac{C_L}{C+k+x}\right]^{\beta}[1-H(C+x)]^n\, h(C)dC. \tag{B4}$$

It is logically straightforward but a bit messy to obtain the NE. Take the derivative of Equation (B4) with respect to x and evaluate it at $x = 0$. As explained further in the proof, any symmetric NE k^* must be a root of the resulting expression, and can be found by Newton's method.

Again, we are especially interested in the case of uniform distributions H and triopoly ($n = 2$). The results are summarized in the following

Theorem B1. *Let H be the uniform distribution on $[C_L, C_U]$. Then for $n+1 = 3$ total investors and every $\beta \in [0, \infty)$, there is a unique symmetric Nash equilibrium $k^*(\beta) > 0$ of the restricted preemption game.*

Equilibrium strategies can be found by numerically solving expression (B6) or, if the cost distribution is uniform, expression (B7). Both are derived below in the proof. The numerical solutions to Equation (B7) for $\beta = 2.25$ and $\beta = 3.00$ are shown in Figure 1 as Nash equilibrium markups for the restricted (1-dimensional) $k^* \approx 7.81$ and $k^* \approx 7.15$ respectively for $\beta = 2.25$ and $\beta = 3.00$. As constant markup rules, the threshold functions are parallel to the zero profit Marshallian threshold. Figure 1 shows that the (restricted) equilibrium constant markups are fairly close to the (unrestricted) BNE markups at the low end of the cost range. However, the constant markups are higher than the BNE markups in the upper three quarters of the cost range, and the divergence increases in cost.

Before deriving k^*, we first analyze monopoly ($n = 0$). With the restricted strategy set, the

[3] We drop "Bayesian" since the restricted strategies are not contingent on cost type.

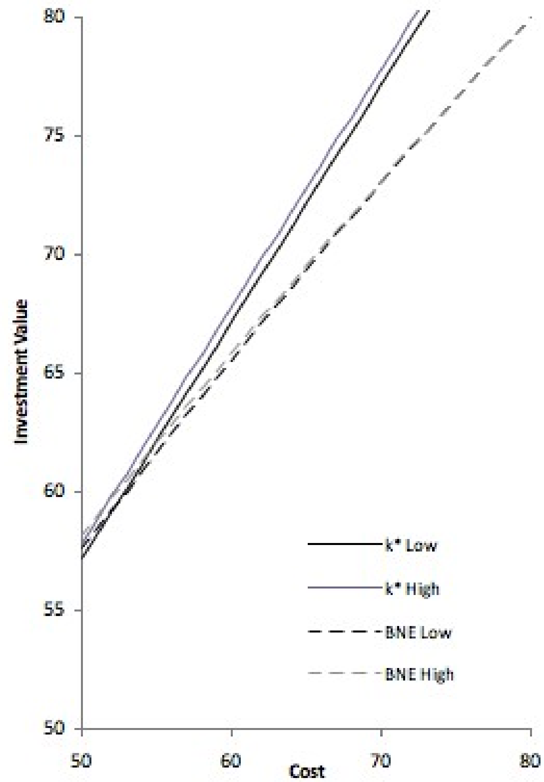

Figure 1: Numerical values of the constant markup equilibrium, k^*, for a parameter set with $\beta = 2.25$ (the High treatment) and a parameter set with $\beta = 3.00$ (the Low treatment). Bayes Nash equilibria for each parameter set are also shown for comparison.

monopolist's value function simplifies to $F_M^R(k, C_i, V) = k[V/(C_i + k)]^\beta$, with expected value

$$E_H F_M^R = kV^\beta \int_{C_L}^{C_U} (C + k)^{-\beta} h(C) dC. \tag{B5}$$

Standard arguments confirm that the value function (B5) has a unique maximum $k_M^* > 0$ as long as $\beta > 1$ (a necessary condition in monopoly, as noted in Section A.1 above).

Proof of Theorem B1. With $n \geq 1$ other investors, we seek a symmetric Nash equilibrium markup k^*. Investor i's objective function is (B4), as noted earlier. To explain, recall that the initial value of the Brownian value is C_L by convention. Thus the integrand is simply the prior probability of "preemption by Nature" or other investors given realized cost C. Of course, the factor outside the integral is simply the profit earned when not preempted. Hence (B4) is the expected profit for constant markup $(k + x)$.

Taking the derivative of Equation (B4) with respect to x, and imposing the incentive condition that (B4) is maximized at $x = 0$, yields the following first order condition:

$$[E_H F]_x = \int_{C_L}^{C_U} (C_i + k)^{-\beta} [1 - H(C_i)]^n h(C_i)\, dC_i - k\beta \int_{C_L}^{C_U} (C_i + k)^{-1-\beta} [1 - H(C_i)]^n h(C_i)\, dC_i - kn \int_{C_L}^{C_U} (C_i + k)^{-\beta} [1 - H(C_i)]^{n-1} [h(C_i)]^2 dC_i = 0. \quad \text{(B6)}$$

A solution of Equation (B6) characterizes the NE k^*, and makes it possible to obtain comparative statics in n, β and H. Again, we are especially interested in the case of uniform distributions H and triopoly ($n = 2$). In this case the equation simplifies to:

$$2(C_U + k)^{3-\beta} - (3-\beta)(2-\beta)(C_U - C_L)^2 (C_L + k)^{1-\beta} - 2(3-\beta)(C_U - C_L)(C_L + k)^{2-\beta} - 2(C_L + k)^{3-\beta} - k(3-\beta)(2-\beta)(1-\beta)(C_U - C_L)^2 (C_L + k)^{-\beta} = 0. \quad \text{(B7)}$$

The solution k^* can be found numerically using Newton's method.

C Product Limit Estimation

Our description of the product limit estimator and log rank test closely mirror the descriptions provided in an appendix of Oprea, Friedman and Anderson (2009). We include it here for completeness.

The product-limit estimator produces an estimate of the distribution function, $F(v_i) = prob(x \leq v_i)$ that takes account of information contained in censored observations to correct censoring bias. Intuitively, a discrete distribution function can be expressed as the product of interval conditional probabilities. By calculating each of these conditional probabilities using only observations that have not been censored at smaller values, unbiased conditional probabilities can be formed. The product of these unbiased interval probabilities for premiums less than or equal to x is the product limit estimate of the distribution function at premium x.

Consider a sample consisting of n observations. In uncensored observations denote v as the investment value and in censored cases denote v as the highest value available before censoring. We can then construct a vector of these values $v = (v_0, v_1, ...v_n)$ ordered so that $i < j$ if and only if $v_i < v_j$. Include in this vector $v_0 = 50$, the lowest value possible at investment.

The product-limit estimate of $F(v_i)$ exploits the fact that the complement of the distribution function can be written as a product of conditional probabilities. Note that $1 - prob(x \geq v_i) = 1 - prob(x \geq v_i | x \geq v_{i-1}) \times prob(x \geq v_{i-1})$. Recursively then,

$$F(v_i) = 1 - \prod_{j=1}^{i} prob(x \geq v_j | x \geq v_{j-1}) \tag{C1}$$

Let c_i denote the number of observations that are censored in $(v_{i-1}, v_i]$ and let d_i denote the number of investments at value v_i. Finally, define $n_i = n_{i-1} - c_{i-1} - d_{i-1}$, the number of subjects available to invest (who have not yet invested nor been censored) at v_i. The product-limit estimate of $p(x \geq v_i | x \geq v_{i-1})$ is the proportion of still-in-sample investors at v_i who do not invest at v_i:

$$\hat{p}(x \geq v_i | x \geq v_{i-1}) = \frac{n_i - d_i}{n_i} \tag{C2}$$

The product-limit estimator is then, following (C1), the cumulative product of these individual conditional probabilities at each v_i

$$\hat{F}(x) = 1 - \prod_{v_i < x} \frac{n_i - d_i}{n_i} \tag{C3}$$

Without censoring (that is when all v_i denote values at investment) it can be shown that $\hat{F}(v_i)$ is simply the empirical distribution function – the proportion of investments which are lower than v_i for each v_i. Kaplan and Meier (1958) show that the product-limit estimator is the maximum likelihood *non-parametric* estimator of the distribution function in environments with censoring problems analogous to ours.

Typically, hypothesis tests comparing product-limit distribution functions are conducted using a log-rank test. Consider two samples, labeled $j = 1, 2$. In what follows we will sometimes pool the two samples and construct the statistics described above, in which case we omit the j subscript. In other cases we'll use statistics described above computed only for pool j in which cases we include the subscript. Thus $n_i = n_{i1} + n_{i2}$, for example.

Now let $d_i = d_{i1} + d_{i2}$ represent the total number of investments made at value v_i. Under the null hypothesis that the two samples are the same at v_i, expected investments in group 1 are $n_{i1} d_i / n_i$ while the actual observed investment is simply d_{i1}. As with many non-parametric techniques, the log-rank test relies on a test statistic based on the difference between these observed and expected statistics. To construct the test statistic, the log-rank test computes the hypergeometric variance for the number of investments at value v_i as

$$s_i^2 = \frac{n_{i1} n_{i2} (n_i - d_i) d_i}{n_i^2 (n_i - 1)} \tag{C4}$$

The test statistic for the log rank test is then

$$z = \frac{\sum_i^n (d_{i1} - \frac{n_{i1} d_i}{n_i})}{\sqrt{\sum_i^n s_i^2}} \tag{C5}$$

which is approximately distributed standard normal under the hypothesis that the hazard rates for the two samples are equal.

D Supplemental Data Analysis

Order Effects in Monopoly Data

In the Low treatment, there is little difference between the early block and the later block. The PL median is 15.10 and 15.18 for the early and late blocks respectively, indicating no ordering effect. In the High treatment, however, there is a significant difference. In the early block the PL median markup is 21.71, dropping to 13.07 following the competition periods. This drop is significant at the one percent level by a log-rank test.

These trends run in the opposite direction of those observed in OFA, where investment values tend to increase over time, a difference that we attribute to contamination from the intervening Competition periods. It is for this reason we omit the final block from the analysis in the body of the paper. However, as we now demonstrate, this omission does not alter the comparative statics reported in the text.

We estimate (14, from the paper) on the feasible subsample for data pooling Monopoly I, Competition and Monopoly II. Results are shown in Table 1. Note that κ is significantly greater than zero. A Wald test confirms that $\kappa + \delta$ is also greater than zero ($p = 0.000$). This confirms Hypothesis 1. Note also that ψ is insignificantly different from zero but, by a Wald test, $\psi + \delta$ is ($p = 0.002$). This confirms Hypothesis 2. Therefore, the cross treatment comparisons (the only ones concerning the Monopoly data) reported in the first two findings are not dependent on excluding the Monopoly II

Variable	Coefficient	Estimate	Standard Error
$Intercept$	γ	5.929	1.559***
$High$	ψ	-0.869	1.854
$Monopoly$	κ	8.664	1.875***
$High \times Monopoly$	δ	13.489	2.6181***

Table 1: Results from nested random effects model (14) conducted on complete data set (including Monopoly II). One, two and three stars designate significance at the ten percent, five percent and one percent levels.

block.

Non Parametric Tests of First Two Hypotheses

In this section we report robustness tests of the first two results using non-parameteric methods.

PL investment value estimates can be compared statistically using a variant of the Mann-Whitney test called the log-rank test; again see Appendix C. That test confirms that the differences between Monopoly and Competition are significant at the one percent level for both High parameters and for Low parameters. As a supplementary test to control for within-subject variance, we also estimate the product-limit mean investment value for each subject under each treatment and market structure. Comparing the populations of individual estimates using Mann-Whitney tests confirms that investment values are higher under Monopoly than under Competition for both Low parameters and High parameters (both with $p = 0.000$). These conform with our first finding.

Product limit investment values are larger under High parameters than Low parameters in the Monopoly block. This difference is significant at the one percent level according to the relevant log-rank test. In Competition, the Low investment values exceed High by about 1-3 points at most percentiles (the opposite direction from that one might expect) but the log-rank test indicates that the difference is insignificant ($p = 0.1090$). Mann-Whitney tests on by-subject PL mean values lead to the same conclusion: values are higher under High parameters under Monopoly ($p = 0.0042$) though not under Competition ($p = 0.2460$). These are similar to the observations collected as our second finding.

E Instructions to Subjects

Instructions were presented to subjects in two parts. Part 1 pertained to the Monopoly treatment and Part 2 to the Competition treatment. Subjects were given Part 1 at the beginning of the session and Part 2 after the completion of period 10, just before the Competition block began. Part 1 of the instructions were complemented with a projected display of the computer interface on a screen.

Part 1

You are about to participate in an experiment in the economics of decision-making. The National Science Foundation and other agencies have provided the funding for this project. If you follow these instructions carefully and make good decisions, you can earn a CONSIDERABLE AMOUNT OF MONEY, which will be PAID TO YOU IN CASH at the end of the experiment.

Your computer screen will display useful information. Remember that the information on your computer screen is PRIVATE. To insure best results for yourself and accurate data for the experimenters, please DO NOT COMMUNICATE with the other participants at any point during the experiment. If you have any questions, or need assistance of any kind, raise your hand and one of the experimenters will come.

In the experiment you will make investment decisions over several rounds. At the end of the last period, you will be paid $5.00, plus the sum of your investment earnings over all rounds.

The Basic Idea. Each round you will decide when (if ever) to seize an investment opportunity. At the beginning of the round you will be assigned a cost, C of investing. The value V of investing will change randomly over time. You earn V-C points if you seize the opportunity before it disappears. If you wait longer, V might go higher, earning you more points. Or V might go lower. The opportunity to invest might evaporate before you seize it, in which case you earn 0 points that round.

Investor Screen Information. Your cost C is shown on your screen as a horizontal red line, as in Figure 1 [Figure 9 here]. The value V of the investment is shown as a jagged green line that scrolls from left to right, with the rightmost tip (the leading edge) representing the current value V. Previous values move left, as on a ticker tape. At the start of each round the value line V starts at 50 and randomly evolves from there.

When you want to invest, press the SPACE BAR.

Other useful messages appear in the window to the right labeled Your Performance. For example, in Figure 1 [Figure 9], that window tells you the round number, your cost, and that you have no competitors for the investment opportunity. You can see messages and results from previous rounds by clicking the Previous button at the top of the window.

The round will continue until the investment opportunity disappears even if you have already invested.

Feedback

After the period is over, you will be shown a chart, reproduced in Figure 1 [Figure 9], repeating your cost, the value you invested at and the number of seconds that elapsed before you invested. The final line in the chart, marked [End] shows you how many seconds the round lasted.

Payment. Points translate into dollars according to a formula written on the board. You will be paid in cash at the end of the experiment for the points earned in all rounds plus the \$5 show-up fee. For example, if the formula is \$0.02 per points in excess of 1000, and if you earn 1682 points, then your cash payment is \$5.00 + \$(1682 1000)*0.02 = \$5.00 + \$13.64 = \$18.64.

Details. In case you want to know, here are a few details of how V unfolds. You can skip these if you prefer to learn just from experience.

- The round is a series of many ticks (e.g., 5 ticks per second).
- Each tick the value V moves randomly up or down by a fixed percentage, e.g., 3%.
- Upticks are slightly more likely than downticks, e.g., each tick is up with probability 51% or down with probability 49%.
- The round ends (the investment evaporates) with a small probability each tick, e.g., $\frac{1}{2}$ of 1%.
- The actual values (for ticks per second, tick size, uptick probability, and evaporation probability) will be written on the board before the experiment begins.
- The value always starts at 50.
- The computer will not allow you to seize the investment opportunity when V is less than C, because that would give you a negative number of points.

Frequently Asked Questions

Q1. Is this some kind of psychology experiment with an agenda you haven't told us?

Answer: No. It is an economics experiment. If we do anything deceptive, or don't pay you cash as described, then you can complain to the campus Human Subjects Committee and we will be in serious trouble. These instructions are meant to clarify how you earn money, and our interest is in seeing how people make investment timing decisions.

Q2. How long does a round last? Is there a minimum or maximum?

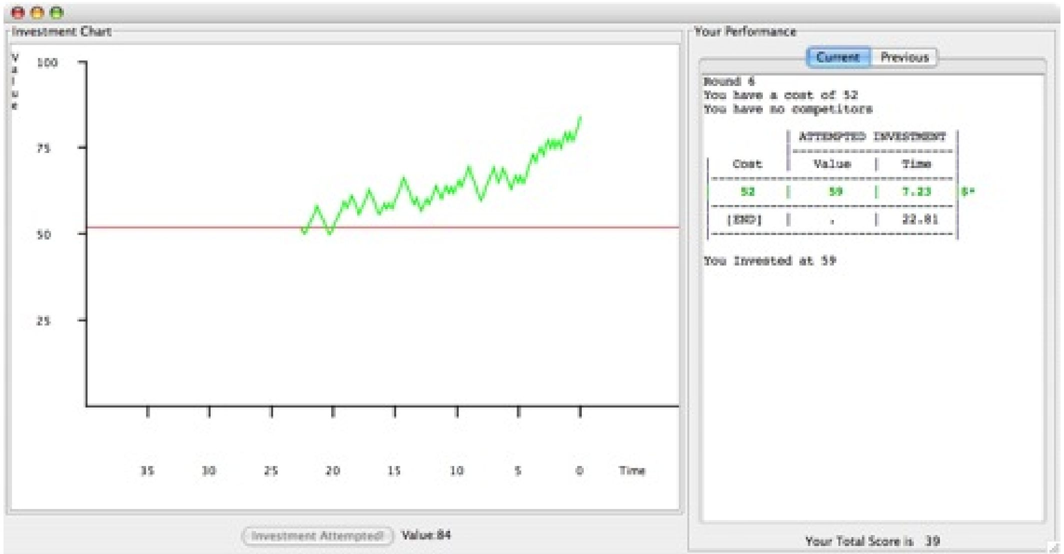

Figure 2: Graphic included in first part of instructions.

Answer: The length of time is random. In the example, the probability is 0.005 that any tick is the last, and there are 5 ticks per second. In this case, the average length of a round is 200 ticks or 40 seconds. Many rounds will last less than the average, and a few will last much longer. Rounds longer than 7 minutes are so unlikely that you probably will never see one. The minimum length is one tick, but it is unlikely you will ever see a round quite that short!

Q3. How many rounds will there be?

Answer: Lots. We arent supposed to say the exact number, but there will be a number of rounds.

Q4. Are there patterns in upticks and downticks?

Answer: No. We've tried very hard to make it random. No matter what the recent history of upticks and downticks, the probability that the next tick is up is always the same (and is written on the board).

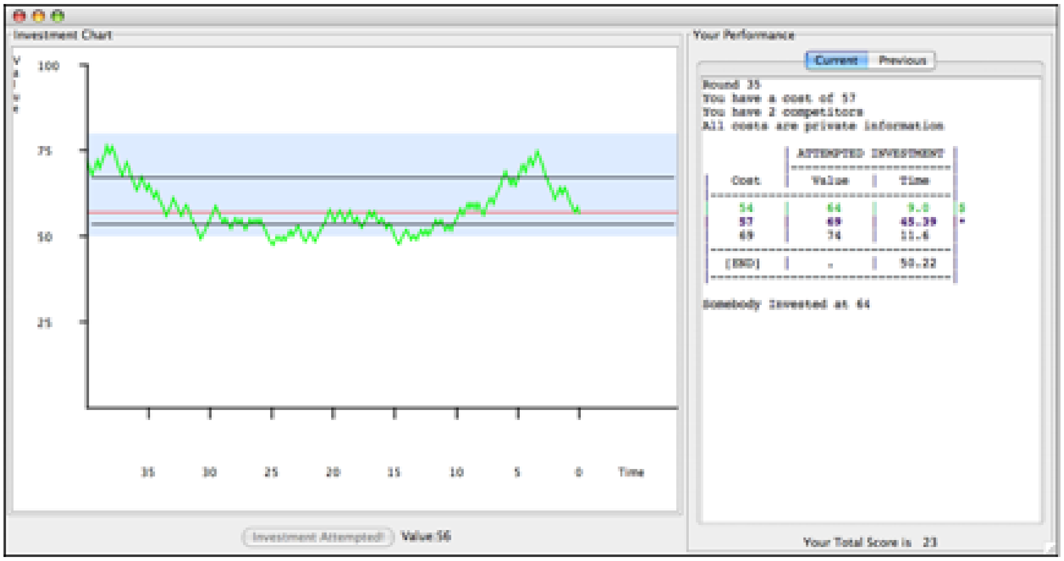

Figure 3: Graphic included in second part of instructions.

Part 2

In this part of the experiment everything will be as before except you will now be grouped each period with 2 other investors. Each investor in your group will secretly decide when to attempt to invest by pressing the space bar. You will not find out who invested when until after the period is over.

Each participant will be assigned a cost each period randomly chosen between 50 and 80 with equal likelihood. This range is shaded in blue on your display. During the period you will see your cost as a red line and will not see the actual costs of other participants.

After the period is over, we will reveal who attempted to invest first, second and third in your group. Whoever attempted to invest first will have successfully invested and will get V-C points. Whoever attempted to invest second and third will fail to invest and will get zero points. This information will be revealed in a table on the right side of the screen when the period is over. The table will have a line for each participant and will be ordered by cost. The participant who successfully invested will be listed in green and will have a dollar sign beside it ($). Your information will be in bold type and will have an asterisk (*) beside it. If you are the successful investor, your information will have both a dollar sign and asterisk beside it and will be in both bold and green.

In the example above, you had a cost of 57 and attempted to invest at a value of 69 after 45.39 seconds. Another participant had a cost of 54 and attempted to invest at a value of 64 after 9 seconds. A third participant had a cost of 69 and attempted to invest at a value of 74 after 11.6

seconds. Because the first participant attempted to invest at the earliest time, he or she successfully invested and will earn 64-54=10 points. Note that this participants information is in green and has a dollar sign by it. You and the other participant earn nothing this period.

The fourth line marked with [End] gives you the number of seconds the period lasted. In the example above, the period lasted a total of 50.22 seconds.

After each period, you will be randomly matched into a new group. You will never be matched in the same group twice.