1 Credit Markets (90 points)

Consider an economy with two consumers who live for two periods. Consumer 1 is very impatient, and therefore places a high weight on utility in the first period. The utility function for the first consumer is

$$u_A(c_1, c_2) = \ln c_1 + \frac{1}{2} \ln c_2.$$  

The second consumer is patient and values both periods equally. His utility function is:

$$u_B(c_1, c_2) = \ln c_1 + \ln c_2.$$  

Both consumers have a fixed income of one unit of consumption good in each period. The consumption good cannot be stored, but the two consumers can trade with each other in the credit market. The interest rate on savings is denoted by $r$.

1. Set up the optimization problem of each consumer and write down the market clearing conditions in the goods market and in the credit market. (20 pts)

   Household A:

   $$\max_{c_1, c_2} \ln c_1 + \frac{1}{2} \ln c_2 \quad s.t. \quad c_1 + c_2 \frac{c_2}{1+r} = 1 + \frac{1}{1+r}$$

   Household B:

   $$\max_{c_1, c_2} \ln c_1 + \ln c_2 \quad s.t. \quad c_1 + \frac{c_2}{1+r} = 1 + \frac{1}{1+r}$$
Market clearing in the consumption goods market:
\[ c_A^1(r) + c_B^1(r) = 2. \]
\[ c_A^2(r) + c_B^2(r) = 2. \]

If savings are denoted by \( s_A^1(r) \) and \( s_B^1(r) \) then market clearing in the credit market is given by
\[ s_A^1(r) + s_B^1(r) = 0. \]

2. Find the consumption of both consumers in the two periods as a function of the interest rate. (25 pts)

The first order conditions are
\[
\begin{align*}
\frac{1}{c_A^1} &= \frac{1 + r}{2} \frac{1}{c_A^2}, \\
\frac{1}{c_B^1} &= (1 + r) \frac{1}{c_B^2}
\end{align*}
\]
and by using budget constraints, one gets
\[
\begin{align*}
c_A^1 + \frac{c_A^2}{1 + r} &= \frac{3}{2} c_A^1 = 1 + \frac{1}{1 + r} \\
c_B^1 + \frac{c_B^2}{1 + r} &= 2 c_B^1 = 1 + \frac{1}{1 + r}.
\end{align*}
\]
Hence
\[
\begin{align*}
c_A^1(r) &= \frac{2}{3} (1 + \frac{1}{1 + r}), \quad c_A^2(r) = \frac{1}{3} (2 + r) \\
c_B^1(r) &= \frac{1}{2} (1 + \frac{1}{1 + r}), \quad c_B^2(r) = \frac{1}{2} (2 + r)
\end{align*}
\]

3. Find the equilibrium interest rate (25 pts).

Market clearing condition \( c_A^1(r) + c_B^1(r) = 2 \) implies that
\[
\begin{align*}
\frac{2}{3} (1 + \frac{1}{1 + r}) + \frac{1}{2} (1 + \frac{1}{1 + r}) &= 2 \\
1 + \frac{1}{1 + r} &= \frac{12}{7} \\
\frac{1}{1 + r} &= \frac{5}{7} \\
r^* &= \frac{2}{5}
\end{align*}
\]
and so the equilibrium interest rate is 40%.
4. Compute the equilibrium savings of both agents. Provide an intuition for the equilibrium pattern of savings. (20 pts)

Consumption in the first period is given by

\[ c_A^* = \frac{2 \frac{12}{7}}{3} = \frac{8}{7} \]
\[ c_B^* = \frac{1 \frac{12}{7}}{2} = \frac{6}{7} \]
\[ c_A^* = \frac{\frac{1 + r^*}{2} c_A^*}{2} = \frac{7}{10} c_A^* = \frac{8}{10} \]
\[ c_B^* = (1 + r^*) c_B^* = \frac{7}{5} c_B^* = \frac{12}{10} \]

Hence savings are given by \( s_A^* = -\frac{1}{7} \) and \( s_B^* = \frac{1}{7} \). The impatient guy wants to consume more today, and so he borrows from the patient guy.

2. Subsidizing Leisure (105 points)

According to the European Union, vacation time is everyone’s right, and the European Union plans to subsidize it. This exercise studies the consequences of a policy that subsidizes leisure.

Suppose that consumers have one unit of time, and they can spend it either by working or by taking leisure. Denote time spent working by \( h \) and leisure by \( l = 1 - h \). Consumers have preferences over consumption \( c \geq 0 \) and leisure \( 1 - h \in [0, 1] \) given by

\[ U(c, l) = c + \ln(1 - h). \]

The government chooses the income tax rate \( t \) and a leisure subsidy \( s \). As a result, the budget constraint of the consumer is

\[ c = w(1 - t)h + s(1 - h), \]

where \( w > 1 \) is the wage rate.

1. Compute the optimal labor supply \( h \) as a function of \( t \) and \( s \). (Note: Assume that the tax system is such that \( w(1 - t) - s \geq 1 \). If you’ve done things correctly, this should yield \( h \in (0, 1) \). ) (25 pts)
\[
\max_h (w(1-t) - s)h + s + \ln(1 - h)
\]
\[
w(1-t) - s = \frac{1}{1-h}
\]
\[
1 - h = \frac{1}{w(1-t) - s}
\]
\[
h = 1 - \frac{1}{w(1-t) - s}.
\]

2. Compute the agent’s utility as a function of \(t\) and \(s\). Show that it is an increasing function of \(s\) and a decreasing function of \(t\). (20 pts)

\[
U = (w(1-t) - s)(1 - \frac{1}{w(1-t) - s}) + s + \ln(\frac{1}{w(1-t) - s})
\]
\[
= w(1-t) - 1 - \ln(w(1-t) - s)
\]

\(U\) is clearly increasing in \(s\). To show that it is decreasing in \(t\), differentiate:

\[
\frac{dU}{dt} = -w + \frac{w}{w(1-t) - s} = -wh < 0
\]

3. In general equilibrium, the government’s budget needs to be balanced and so taxes and subsidies must be related. In particular, the total amount of taxes collected is equal to the total amount of leisure subsidies paid:

\[
wth = s(1-h). \tag{1}
\]

Show, using your solution for the optimal labor supply, that there exists a Laffer curve in this economy in a sense that leisure subsidies must be zero if either \(t = 0\) or if \(t = 1 - \frac{1}{w}\). (20 pts)

\[
s = wt \frac{h}{1-h}
\]
\[
= wt(w(1-t) - s - 1)
\]

Hence

\[
s = \frac{wt}{1 + wt}(w(1-t) - 1).
\]

The right-hand side is zero if \(t = 0\), or if \(w(1-t) - 1 = 0\), which yields \(t = 1 - \frac{1}{w}\).
4. Use the government’s budget constraint (1) to eliminate the leisure subsidy \( s \) from your solution for the optimal labor supply, and to express the labor supply only as a function of the tax rate \( t \). (20 pts)

\[ s = wt \frac{h}{1 - h}. \]

Hence

\[
1 - h = \frac{1}{w(1 - t - t \frac{h}{1-h})}
\]
\[
1 - h = \frac{1}{w(1 - \frac{t}{1-h})}
\]
\[
1 - h = \frac{1 - h}{w(1 - h - t)}
\]

\[ h = 1 - t - \frac{1}{w}. \]

5. Express now the agent’s utility as a function of \( t \) only, and show that the agent’s utility is decreasing in \( t \). Conclude that leisure subsidies can never increase welfare. (20 pts)

Consumption is, using (1),

\[ c = w(1 - t)h + s(1 - h) = wh. \]

Hence the lifetime utility is

\[ U = wh + \ln(1 - h) \]
\[ = w(1 - t) - 1 + \ln(t + \frac{1}{w}). \]

Differentiating, we get

\[ \frac{dU}{dt} = -w + \frac{w}{1 + tw} < 0. \]