Economics 204b Marek Kapička
Spring 2009

Problem Set 5

due Monday June 1

You can work in pairs. In addition to a printed copy of your solution, please send your MATLAB files to Jesse.

Linear-Quadratic Approximation

In this problem set you will numerically implement the linear-quadratic approximation of a Real Business Cycle model. We are interested in solving the following problem:

$$V(k,\omega) = \max_{n,i}\{(1-\mu)\ln(e^{\omega}k^{\alpha}n^{1-\alpha} - i) + \mu\ln(1-n) + \hat{\beta}E[V(k',\omega')|\omega]\}$$

s.t.

$$\begin{aligned} k' &= \frac{1-\delta}{(1+\eta)(1+\gamma)}k + \frac{1}{(1+\eta)(1+\gamma)}i \\ \omega' &= \rho\omega + \nu\varepsilon \qquad \varepsilon \sim N(0,1), \end{aligned}$$

where k is capital per effective labor, n is labor supply per person, i is investment per effective labor and $\hat{\beta} = \beta(1+\eta)$. The steady state values of $\bar{c} = \bar{k}^{\alpha}\bar{n}^{1-\alpha} - \bar{i}, \bar{k}, \bar{n}$ are given by the first order and envelope conditions. They can be reduced to the following system of equations:

$$\begin{aligned} \frac{1+\gamma}{\beta} &= \alpha(\frac{\bar{n}}{\bar{k}})^{1-\alpha} + 1 - \delta \\ (1+\eta)(1+\gamma) &= (\frac{\bar{n}}{\bar{k}})^{1-\alpha} + 1 - \delta - \frac{\bar{c}}{\bar{k}} \\ \frac{\bar{c}}{\bar{k}}\frac{\bar{n}}{1-\bar{n}} &= (1-\alpha)\frac{1-\mu}{\mu}(\frac{\bar{n}}{\bar{k}})^{1-\alpha}. \end{aligned}$$

1. Set $\mu = 0.36$, $\alpha = 0.4$, $\eta = 0.003$, $\beta = 0.987$, $\gamma = 0.0039$ $\rho = 0.95$, $\delta = 0.012$, $\nu = 0.007$. Use MATLAB to compute the steady state values of $\bar{k}, \bar{n}, \bar{i}$. [Hint: compute first $\frac{\bar{n}}{\bar{k}}$ using the first equation. Then compute $\frac{\bar{c}}{\bar{k}}$ using the second one and, given that, $\bar{n}$ using the third one. Then compute $\bar{k}$, $\bar{c}$ and finally $\bar{i}$].

We will now compute the linear-quadratic approximation around the steady state.

2. Let $y = \begin{bmatrix} \omega & k & n & i \end{bmatrix}^T$ and $r(y) = (1-\mu)\ln(e^{\omega}k^{\alpha}n^{1-\alpha} - i) + \mu \ln(1-n)$. The second order approximation of the objective function is given by a function $\tilde{r}(y)$ satisfying

$$\tilde{r}(y) = r(\bar{y}) + \bar{J}^T(y - \bar{y}) + \frac{1}{2}(y - \bar{y})^T \bar{H}(y - \bar{y}).$$

where

$$\bar{J} = \begin{bmatrix} r_\omega(\bar{y}) & r_k(\bar{y}) & r_n(\bar{y}) & r_i(\bar{y}) \end{bmatrix}^T$$

and

$$\bar{H} = \begin{bmatrix} r_{\omega\omega}(\bar{y}) & r_{\omega k}(\bar{y}) & r_{\omega n}(\bar{y}) & r_{\omega i}(\bar{y}) \\ r_{\omega k}(\bar{y}) & r_{kk}(\bar{y}) & r_{kn}(\bar{y}) & r_{ki}(\bar{y}) \\ r_{\omega n}(\bar{y}) & r_{kn}(\bar{y}) & r_{nn}(\bar{y}) & r_{ni}(\bar{y}) \\ r_{\omega i}(\bar{y}) & r_{ki}(\bar{y}) & r_{ni}(\bar{y}) & r_{ii}(\bar{y}) \end{bmatrix}.$$

(a) Compute the numerical values of $\bar{J}$ and $\bar{H}$. [Hint: you can either use the Symbolic math toolbox in MATLAB to compute the derivatives or you can do them analytically.]

(b) Use the matrices $\bar{J}$ and $\bar{H}$ to compute the matrix Q that satisfies

$$\tilde{r}(y) = \begin{bmatrix} 1 & y^T \end{bmatrix} Q \begin{bmatrix} 1 \\ y \end{bmatrix}.$$

(c) Let $x = \begin{bmatrix} 1 & \omega & k \end{bmatrix}^T$ and $u = \begin{bmatrix} n & i \end{bmatrix}^T$. Compute Q_{xx}, Q_{xu}, Q_{uu} satisfying

$$Q = \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{xu}^T & Q_{uu} \end{bmatrix}.$$

(d) Compute the matrices A and B such that the constraints satisfy (if there is no uncertainty)

$$x' = Ax + Bu. \tag{1}$$

(e) Start with a zero 3×3 symmetric matrix P_1 and iterate on the Riccati difference equation

$$\begin{aligned} P_{j+1} &= Q_{xx} + \hat{\beta}A^T P_j A \\ &\quad -(\hat{\beta}B^T P_j A + Q_{xu}^T)^T (Q_{uu} + \hat{\beta}B^T P_j B)^{-1}(\hat{\beta}B^T P_j A + Q_{xu}^T) \end{aligned}$$

until $\| P_{j+1} - P_j \| \leq 1e-6$.

(f) The optimal labor supply and investment is given by $u = Fx$ where the 2×3 matrix F is given by

$$F = -(Q_{uu} + \hat{\beta}B^T P B)^{-1}(\hat{\beta}B^T P A + Q_{xu}^T).$$

Compute F.

In the remaining part of the problem set you will use your approximation to simulate the behavior of our RBC model.

3. Suppose that we start in the steady state. Consider an initial shock $\varepsilon_1 = 1$. Use the matrix F to compute the impulse response functions (percentage deviations from the steady state values) for labor supply per person, investment per effective labor, consumption per effective labor, output per effective labor, and capital stock per effective labor for 100 periods. [Hint: to compute the impulse response function for capital stock use the law of motion for x, $x_{t+1} = (A + BF)x_t + C\varepsilon_t$, where $x_1 = \bar{x}$, $\varepsilon_1 = 1$, and $\varepsilon_t = 0$, $t > 1$. To compute the impulse response functions for consumption and output, use the resource constraint and the production function]. Plot all the impulse response functions on one page using the MATLAB function `subplot`.

4. Simulate a time series for the shock ω for 100 periods. Use the optimal policy functions to simulate time series for percentage deviations of labor supply per person, investment per effective labor, consumption per effective labor, output per effective labor and capital stock per effective labor from their steady state values. Report the volatility of each series and cross-correlation with output. Plot your results.