Economics 204b Marek Kapicka
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# Problem Set 2

## 1 Investment with convex costs

Solve the exercise 5.9 in Stokey and Lucas

## 2 A stochastic cake-eating problem

Consider an individual with preferences

$$u(c_0, c_1, ...) = \sum_{t=0}^{\infty} \beta^t \frac{z_t c_t^\theta}{\theta}, \qquad \rho > 0, \quad 0 < \theta < 1,$$

where the preference shock $\{z_t\}$ is i.i.d. and takes values in a finite set $Z$. The individual must choose an optimal path for consuming out of wealth $A$. The interest rate is $r > 0$, so the law of motion for wealth is

$$A_{t+1} = (1 + r)(A_t - c_t).$$

1. Formulate the optimal consumption problem as a dynamic program.

   Assume that $r < \beta - 1$ so that wealth can be expected to decline over time and let $X = [0, \bar{A}]$ be the state space for wealth. With this parameter restriction and this state space, the problem is bounded.

   A standard method for showing that the value function and/or the policy function has a certain property is to show that the operator $T$ preserves the property of interest. This line of argument can be used to show that the value function is homogeneous of degree $\theta$ in wealth and that the policy function is homogeneous of degree one in wealth, and that both are increasing in $z$. Let $H_\theta$ be the family of candidate value functions that have the form $w(A, z) = \phi(z)A^\theta$.

2. Show that the operator $T$ defined by the right hand side of the Bellman equation maps $H_\theta$ into itself. That is, $w \in H_\theta$ implies $Tw \in H_\theta$. (Hint: show that if $w$ has this form, the optimal policy for the current period has the form $c = g(A, z) = \gamma(z)A$.)

3. Show that if $w$ is increasing in $z$ then so is $Tw$ and so is the associated optimal policy function $g(A, z) = \gamma(z)A$.

4. Suppose that the shocks $\{z_t\}$ are first-order Markov instead of i.i.d. Explain briefly whether or not you would expect the results in part (d) to hold.