Economics 204b — Marek Kapicka
Spring 2009

Problem Set 1

1 A cake-eating problem

Consider an individual with preferences

$$u(c_0, c_1, ...) = \sum_{t=0}^{\infty} (\frac{1}{1+\rho})^t \ln(c_t), \qquad \rho > 0,$$

who must choose an optimal path for consuming a cake of fixed size A. The cake does not grow or decay, so the constraint on consumption is simply

$$\sum_{t=0}^{\infty} c_t \leq A.$$

Formulate the optimal consumption problem as a dynamic program. Use the "guess and verify" method (See RMT section 3.1.1) to find the value function and the optimal policy function.

2 Computing the optimal growth model with dynamic programming

Only in very special cases can dynamic programming be solved using paper and pencil methods. This problem is intended to get you started in using the computer to solve more general problems. You are asked to use the iteration on the value function method to compute the value function and the associated policy function for the optimal growth model with arbitrary depreciation.

The production technology in the economy is given by a production function $F(k) = k^\alpha$, where k is the capital stock used to produced output y. Given the initial capital stock k_0, the agent's problem is

$$\begin{aligned} \max_{\{c_t,k_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}-1}{1-\sigma} \qquad (1) \\ \text{s.t.} \qquad k_{t+1} &= k_t^{\alpha} - c_t + (1-\delta)k_t, \qquad t = 0,1,2,.. \end{aligned}$$

where $0 < \beta < 1$ is the discount factor.

We know that the maximized value of the objective function in (1) depends on k_0 only. Call it $v^*(k_0)$. We know that this value function satisfies the Bellman Equation:

$$\begin{aligned} v^*(k) &= \max_{c,k'} [\frac{c^{1-\sigma}-1}{1-\sigma} + \beta v^*(k')] \qquad (2) \\ \text{s.t.} \qquad k' &= k^{\alpha} - c + (1-\delta)k. \end{aligned}$$

Unfortunately, just by looking at (2) it is hard to say much about v^*. But the functional equation can also be used to define an algorithm for computing v^* numerically. In particular for any function w define another function Tw by

$$\begin{aligned} Tw(k) &= \max_{c,k'} [\frac{c^{1-\sigma}-1}{1-\sigma} + \beta w(k')] \qquad (3) \\ \text{s.t.} \qquad k' &= k^{\alpha} - c + (1-\delta)k. \end{aligned}$$

Then for any initial function v_0, we can define a sequence of functions $\{v_n\}_{n\geq 0}$ recursively by the formula $v_{n+1} = Tv_n$.

The first part of this exercise will illustrate that this procedure converges: $\lim_{n\to\infty} v_n = v^*$ for any initial function v_0. It will also illustrate that a thoughtful choice of v_0 makes the procedure converge more quickly. (This will be shown by counting the number of iterations. The system is so simple that the computation time is very short even if poor guess is used.]

For all the computations use the parameter values $\alpha = \frac{1}{3}$, $\delta = 0.04$, $\sigma = 0.5$, $\beta = 0.96$ and $k = [5, 10]$. Use a grid of 1500 equally spaced points and a convergence criterion of $\varepsilon = 0.0001$.

1. Let $v_0(k_0) = 0$ all k and compute the sequence $\{v_n\}$ until the convergence criterion

   $$|v_N(k) - v_{N-1}(k)| < \varepsilon, \qquad \text{all } k$$

   is satisfied. (i.e. the maximal difference between successive approximations is smaller than ε. How many iterations are required? Plot the calculated value and policy functions, $v_N(k)$, $c_N(k)$ and $k'_N(k)$.

2. What are the steady state levels of k^{ss}, c^{ss} for the problem in (1)? Let $\eta = \frac{c^{ss}}{F(k^{ss})}$ be the steady state fraction of output that is consumed. Suppose the agent follows the feasible strategy of simply consuming a constant fraction of output η each period, so

$$\hat{c}(k) = \eta k^{\alpha}, \text{ all } k.$$

   What is the discounted lifetime utility, call it $\hat{v}_0(k)$, of an agent who follows this rule?

   Now use $\hat{v}_0(k)$ as a starting point for iterative procedure. Use the same convergence criterion as before to stop. How many iterations does it take to converge? Plot the new calculated value function. Is it the same as in part 1? Compare the value function v_N with the guess $\hat{v}_0$ to see how good it was as a guess.

3. Holding the other parameters constant, compute the value and policy functions for an agent with $\sigma = 5$. Let $k_0 = 5$, and plot the time paths for k_t and c_t for the two types of agents: with $\sigma = 0.5$ and $\sigma = 5$. How does σ affect the paths?

You can work on this problem in pairs (but not on the cake-eating problem). Please send a copy of your MATLAB program to Jesse.