Arrow-Debreu Equilibria
Recursive Competitive Equilibria

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April 27, 2009

Plan for today

- A note on asset pricing
- First and Second Welfare Theorems
- Recursive Competitive Equilibria

1. Arrow-Debreu Equilibrium

Characterization: Asset pricing

- For each state z^t there is one Arrow-Debreu asset, with price determined by

$$q_t(z^t) = \beta^t \frac{U'(c_t(z^t))}{U'(c_0(z_0))} \Pi(z^t|z_0)$$

- Advantage of the A-D trading mechanism: One can use "no arbitrage" argument to price any other asset

Example

(risk free bond) An asset that delivers 1 unit of consumption in period 1 regardless of the state:

Example

(riskless console) An asset that pays 1 unit of consumption forever

Example

(stock price) An asset that pays dividends $d_t(z^t)$

2. Pareto Optimum

- The social planner solves

$$\max_{\{c,k\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U(c_t(z^t)) \Pi(z^t | z_0)$$

s.t.

$$\begin{aligned} c_t(z^t) + k_{t+1}(z^t) \quad &\leq \quad z_t F_k[k_t(z^t), 1] + (1-\delta) k_t(z^{t-1}) \\ & \quad\quad k_0 \text{ given} \end{aligned}$$

First Welfare Theorem

Theorem

If (c, k^s, y, k^d, n^d) *are competitive equilibrium allocations then they are Pareto optimal*

Proof.

Suppose that $(\tilde{c}, \tilde{k}^s)$ is a feasible allocation that yields higher expected utility. Then $\tilde{P} > P$, otherwise it would be chosen by the households. Because $\tilde{P} > P$, (y, k^d, n^d) was not a profit maximizing allocation for the firm, a contradiction. □

First Welfare Theorem

General Result

- All we need for FWT to hold is local nonsatiation of preferences.

Second Welfare Theorem

Theorem

Let $(c^*, k^{*s}, y^*, k^{*d}, n^{*d})$ *be a Pareto optimal allocation. Then there exist prices* (q, w, r) *such that* (q, w, r) *and* $(c^*, k^{*s}, y^*, k^{*d}, n^{*d})$ *constitute a competitive equilibrium.*

Proof.

(sketch) Find a candidate price system (q, w, r) . Verify that the candidate price system together with the Pareto optimal allocations constitutes a competitive equilibrium.
Candidate Prices:

$$
\begin{aligned}
q_t(z^t) &= \beta^t \frac{U'(c_t^*(z^t))}{U'(c_0^*(z_0))} \Pi(z^t|z_0) \\
w_t(z^t) &= z_t F_n[k_t^{*d}(z^{t-1}), n_t^{*d}(z^t)] \\
r_t(z^t) &= z_t F_k[k_t^{*d}(z^{t-1}), n_t^{*d}(z^t)].
\end{aligned}
$$

□

Second Welfare Theorem

Proof.

(sketch) For these prices, the first order conditions are satisfied for $(c^*, k^{*s}, y^*, k^{*d}, n^{*d})$. Since F, U are both strictly concave and differentiable and the Pareto optimal allocation satisfies the transversality condition, the first order conditions are sufficient. □

Second Welfare Theorem

General Result

1. Strict concavity and continuity of the utility function, concavity of the production set
2. (In infinite dimensional spaces) Existence of an interior point in the production set

- Differentiability or boundedness of the production function is not required.

Recursive Competitive Equilibrium

- K : **economywide** capital stock
  - Prices of capital and labor can be expressed as a function of z and K only:

$$\begin{aligned} w(K,z) &= zF_n(K,1) \\ r(K,z) &= zF_k(K,1). \end{aligned}$$

  - Assume that the household expects a law of motion for the economywide capital to be

$$K' = H(K,z)$$

- k : **household** capital stock
  - In equilibrium, we will have $k = K$ but in a household problem they must allowed to be different!

Recursive Competitive Equilibrium

Household and firm problem

- Household's value function: $v(k, K, z)$:

$$\begin{aligned} v(k, K, z) &= \max_{c,k'} \{U(c) + \beta \sum_{z' \in Z} v(k', K', z') \pi(z'|z)\} \\ & \quad s.t. \\ c + k' &\leq [r(K, z) + 1 - \delta]k + w(K, z) \\ K' &= H(K, z) \\ c &\geq 0, k' \geq 0 \end{aligned}$$

  - Optimal policy functions $c(k, K, z)$ and $h(k, K, z)$.

- Firm: Maximize profits by setting

$$\begin{aligned} zF_n(K, 1) &= w(K, z) \\ zF_k(K, 1) &= r(K, z) \end{aligned}$$

Recursive Competitive Equilibrium

Definition

Recursive Competitive Equilibrium is given by

1. Value function $v(k, K, z)$ and the optimal policy functions $c(k, K, z)$ and $h(k, K, z)$
2. Law of motion for the economywide capital $H(K, z)$
3. Price functions $w(K, z)$ and $r(K, z)$

such that

a. $c(k, K, z)$, $h(k, K, z)$ and $v(k, K, z)$ solve the household problem, taking K as given

b. $zF_n(K, 1) = w(K, z)$ and $zF_k(K, 1) = r(K, z)$

c. Aggregate and individual decision rules are consistent: $h(K, K, z) = H(K, z)$

d. Markets clear:

$$c(K, K, z) + h(K, K, z) = zF(K, 1) + (1 - \delta)K$$

Recursive Pareto Optimum

- Social Planner's problem

$$\begin{aligned} V(K,z) &= \max_{C,K'} \{U(C) + \beta \sum_{z' \in Z} V(K',z')\pi(z'|z)\} \\ & s.t. \\ C + K' &= zF(K,1) + (1-\delta)K \end{aligned}$$

- Optimal policy functions $C(K,z)$, $K'(K,z)$
- We want to show versions of FWT and SWT:

$$\begin{aligned} V(K,z) &= v(K,K,z) \\ K'(K,z) &= H(K,z) = h(K,K,z) \end{aligned}$$