Arrow-Debreu Equilibria

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Competitive Equilibria and Pareto Optima

Introduction

$$\begin{tikzcd}
\textbf{PO} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow, "\text{FWT}" near start, "\text{SWT}" near end] & \textbf{RPO} \arrow[d, Leftrightarrow] \\
\textbf{CE} \arrow[r, Leftrightarrow] & \textbf{RCE}
\end{tikzcd}$$

1. Arrow-Debreu Equilibrium

- We will
    - Look at the trading arrangements
    - Look at the firm's problem
    - Look at the household's problem
    - Define the Arrow-Debreu Equilibrium
    - Look at its properties
        - Asset Pricing

1. Arrow-Debreu Equilibrium

Prices

- All the trades are determined at time 0
- Future prices and trades are contingent on the history of shocks $z^t = (z_0, z_1, ...z_t)$
- **Complete markets**: for each z^t there exists a market for consumption goods
- Prices:
  - $q_t(z^t)$: Price of one unit of consumption in state z^t, in terms of time zero consumption
  - $w_t(z^t)$: Price of one unit of labor in state z^t in terms of state z^t consumption
  - $r_t(z^t)$: Rental price of one unit of capital in state z^t in terms of state z^t consumption

1. Arrow-Debreu Equilibrium

Firm's Problem

- Firms make decision about
  - Capital demand $k_t^d(z^t)$
  - Labor demand $n_t^d(z^t)$
  - Output $y_t(z^t)$
- Firms maximize profits

$$\begin{aligned} P &= \max_{\{k^d, n^d, y\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)\left[y_t(z^t) - r_t(z^t) k_t^d(z^t) - w_t(z^t) n_t^d(z^t)\right] \\ & s.t. \qquad y_t(z^t) \leq z_t F\left[k_t^d(z^t), n_t^d(z^t)\right] \end{aligned}$$

1. Arrow-Debreu Equilibrium

Household Problem

- Households decide about
  - Consumption $c_t(z^t)$
  - Capital supply $k_t^s(z^t)$
  - Labor supply $n_t^s(z^t)$
  - Investment $i_t(z^t)$
  - Capital holdings $x_{t+1}(z^t)$

  given the initial capital holdings x_0.

1. Arrow-Debreu Equilibrium

Household Problem

- Households maximize utility

$$\max_{\{k^s,n^s,c,i,x\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U(c_t(z^t)) \Pi(z^t|z_0)$$

s.t.

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[c_t(z^t)+i_t(z^t)] \leq \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[r_t(z^t)k_t^s(z^t)+w_t(z^t)n_t^s(z^t)]+P$$

$$\begin{aligned} x_{t+1}(z^t) &= (1-\delta)x_t(z^{t-1}) + i_t(z^t) \\ 0 &\leq n_t^s(z^t) \leq 1 \\ 0 &\leq k_t^s(z^t) \leq x_t(z^{t-1}) \\ c_t(z^t) &\geq 0 \\ x_{t+1}(z^t) &\geq 0 \\ x_0 & \text{ given} \end{aligned}$$

Definition

Arrow-Debreu Equilibrium is given by

1. a set of prices (q, w, r)
2. an allocation
    - 2.1 (y, k^d, n^d) for the firm
    - 2.2 (c, k^s, n^s, x) for the household

such that

1. (y, k^d, n^d) solves the firm's problem given (q, w, r)
2. (c, k^s, n^s, x) solves the household's problem given (q, w, r)
3. markets clear: for all t, z^t,

$$\begin{aligned} c_t(z^t) + i_t(z^t) &= y_t(z^t) && \text{(consumption goods mkt)} \\ n_t^s(z^t) &= n_t^d(z^t) && \text{(labor mkt)} \\ k_t^s(z^t) &= k_t^d(z^t) && \text{(capital mkt)} \end{aligned}$$

1. Arrow-Debreu Equilibrium

Characterization

1a. First order conditions to the firm's problem: for all t, z^t,

$$\begin{aligned} z_t F_k[k_t^d(z^t), n_t^d(z^t)] &= r_t(z^t) > 0 \\ z_t F_n[k_t^d(z^t), n_t^d(z^t)] &= w_t(z^t) > 0 \end{aligned}$$

   - Factor prices are always strictly positive

1b. Since utility is strictly increasing, Arrow-Debreu prices are also always strictly positive: $q_t(z^t) > 0$ all t, z^t.

2. Assume that F is CRS: $F(\lambda k, \lambda n) = \lambda F(k, n)$. Then $P = 0$.

1. Arrow-Debreu Equilibrium

Characterization

3. Since factor prices are strictly positive (by 1a), we have

$$\begin{aligned} k_t^s(z^t) &= x_t(z^{t-1}) \\ n_t^s(z^t) &= 1 \end{aligned}$$

The household problem becomes

$$\max_{\{k^s, n^s, c, i, x\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U(c_t(z^t)) \Pi(z^t | z_0)$$

s.t.

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[c_t(z^t) + i_t(z^t)] \leq \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[(r_t(z^t) + 1 - \delta) k_t^s(z^{t-1}) + w_t(z^t)]$$

$$c_t(z^t) \geq 0, \; x_{t+1}(z^t) \geq 0, \; k_0 \text{ given}$$

1. Arrow-Debreu Equilibrium

Characterization

4. First order conditions for the household

$$\begin{aligned} \beta^t U'(c_t(z^t))\Pi(z^t|z_0) &= \lambda q_t(z^t) \\ \lambda[(r_t(z^t)+1-\delta)q_{t+1}(z^{t+1})-q_t(z^t)] &\leq 0 \\ &= \text{ if } k_{t+1}(z^t)>0 \end{aligned}$$

Pricing kernel:

$$q_t(z^t)=\beta^t\frac{U'(c_t(z^t))}{U'(c_0(z_0))}\Pi(z^t|z_0)$$

Price of an asset that delivers 1 unit of consumption in state z^t.

1. Arrow-Debreu Equilibrium

Characterization: Asset pricing

- For each state z^t there is one Arrow-Debreu asset, with price determined by

$$q_t(z^t) = \beta^t \frac{U'(c_t(z^t))}{U'(c_0(z_0))} \Pi(z^t | z_0)$$

- Advantage of the A-D trading mechanism: One can use "no arbitrage" argument to price any other asset

Example

(risk free bond) An asset that delivers 1 unit of consumption in period 1 regardless of the state:

Example

(riskless console) An asset that pays 1 unit of consumption forever

Example

(stock price) An asset that pays dividends $d_t(z^t)$

2. Pareto Optimum

- The social planner solves

$$\max_{\{c,k\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U(c_t(z^t)) \Pi(z^t | z_0)$$

s.t.

$$\begin{aligned} c_t(z^t) + k_{t+1}(z^t) &\leq z_t F_k[k_t(z^t), 1] + (1-\delta) k_t(z^{t-1}) \\ & k_0 \text{ given} \end{aligned}$$

First Welfare Theorem

Theorem

If (c, k^s, y, k^d, n^d) *are competitive equilibrium allocations then they are Pareto optimal*

Proof.

Suppose that $(\tilde{c}, \tilde{k}^s)$ is a feasible allocation that yields higher expected utility. Then $\tilde{P} > P$, otherwise it would be chosen by the households. Because $\tilde{P} > P$, (y, k^d, n^d) was not a profit maximizing allocation for the firm, a contradiction. ☐

First Welfare Theorem

General Result

- All we need for FWT to hold is local nonsatiation of preferences.

Second Welfare Theorem

Theorem

Let $(c^*, k^{*s}, y^*, k^{*d}, n^{*d})$ *be a Pareto optimal allocation. Then there exist prices* (q, w, r) *such that* (q, w, r) *and* $(c^*, k^{*s}, y^*, k^{*d}, n^{*d})$ *constitute a competitive equilibrium.*

Proof.

(sketch) Find a candidate price system (q, w, r) . Verify that the candidate price system together with the Pareto optimal allocations constitutes a competitive equilibrium.

Candidate Prices:

$$\begin{aligned} q_t(z^t) &= \beta^t \frac{U'(c_t^*(z^t))}{U'(c_0^*(z_0))} \Pi(z^t|z_0) \\ w_t(z^t) &= z_t F_n[k_t^{*d}(z^{t-1}), n_t^{*d}(z^t)] \\ r_t(z^t) &= z_t F_k[k_t^{*d}(z^{t-1}), n_t^{*d}(z^t)]. \end{aligned}$$

□

Second Welfare Theorem

Proof.

(sketch) For these prices, the first order conditions are satisfied for $(c^*, k^{*s}, y^*, k^{*d}, n^{*d})$. Since F, U are both strictly concave and differentiable and the Pareto optimal allocation satisfies the transversality condition, the first order conditions are sufficient. ☐

Second Welfare Theorem

General Result

1. Strict concavity and continuity of the utility function, concavity of the production set
2. (In infinite dimensional spaces) Existence of an interior point in the production set

- Differentiability or boundedness of the production function is not required.