# Stochastic Dynamic Programming
# Arrow-Debreu Equilibria

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April 20, 2009

# Today

- We will
    1. Look at Stochastic Dynamic Programming
    2. Look at Arrow-Debreu Equilibria

# 7.2. Stochastic Dynamic Programming

## Optimal Growth Problem

- The Sequence Problem

$$v^*(k_0, z_0) = \max_{\{k_{t+1}(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U[z_t f(k_t(z^{t-1})) - k_{t+1}(z^t)]\Pi(z^t|z_0)$$

$$s.t. 0 \leq k_{t+1}(z^t) \leq z_t f((z^{t-1})) \qquad k_0, z_0 \text{ given}$$

-
  - The optimal capital stock $k_{t+1}(z^t)$ and consumption $c_t(z^t)$ are now indexed by history of shocks $z^t$
- The Bellman Equation

$$v(k, z) = \max_{0 \leq y \leq zf(k)} U[zf(k) - y] + \beta \sum_{z' \in Z} v(y, z')\pi(z'|z)$$

  - The optimal policy function $g(k, z)$ now depends on the current shock $z$.

- We will show that the value function in the sequence problem $v^*(k, z)$ satisfies the Bellman Equation
- One can also show that under a certain boundedness condition, the solution to the Bellman Equation $v(k, z)$ satisfies the sequence problem.

# 7.2. Stochastic Dynamic Programming

## A solution to (SP) satisfies (FE)

$$
\begin{aligned}
v^*(k_0, z_0) &= \max_{\{0 \le k_{t+1}(z^t) \le z_t f((z^{t-1}))\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U[c_t(z^t)] \Pi(z^t | z_0) \\
&= \max_{\{0 \le k_{t+1}(z^t) \le z_t f k_t((z^{t-1}))\}_{t=0}^{\infty}} \{ U[z_0 f(k_0) - k_1(z_0)] \\
&+ \beta \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^{t-1} U[c_t(z^t)] \Pi(z^t | z_0) \} \\
&= \max_{\{0 \le k_1(z) \le z_0 f(k_0)\}_{t=0}^{\infty}} \{ U[z_0 f(k_0) - k_1(z_0)] \\
&+ \beta \max_{\{0 \le k_{t+1}(z^t) \le z_t f((z^{t-1}))\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \sum_{z^t \in Z^t} \beta^{t-1} U[c_t(z^t)] \Pi(z^t | z_0) \}
\end{aligned}
$$

# 7.2. Stochastic Dynamic Programming

## A solution to (SP) satisfies (FE)

Since $\Pi(z^t|z_0) = \Pi(z^t|z_1)\pi(z_1|z_0)$,

$$v^*(k_0, z_0) = \max_{\{0 \le k_1(z) \le z_0 f(k_0)\}_{t=0}^{\infty}} \{U[z_0 f(k_0) - k_1(z_0)]$$

$$+ \beta \sum_{z_1 \in Z} \{ \max_{\{\cdot\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \sum_{z^t|z_1 \in Z^{t-1}} \beta^{t-1} U[c_t(z^t)] \Pi(z^t|z_1) \} \pi(z_1|z_0) \}$$

$$= \max_{\{0 \le k_1(z) \le z_0 f(k_0)\}_{t=0}^{\infty}} \{U[z_0 f(k_0) - k_1(z_0)] + \beta \sum_{z_1 \in Z} v^*(k_1, z_1) \pi(z_1|z_0)$$

# 7.3. Stochastic Dynamic Programming

## General Setup

- $z$ can affect
  - the correspondence $\Gamma$
  - the objective function $F$
- The Sequence Problem

$$v^*(x_0, z_0) = \max_{\{k_{t+1}(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t F[x_t(z^{t-1}), x_{t+1}(z^t), z_t] \Pi(z^t | z_0)$$

$$s.t. \; x_{t+1}(z^t) \in \Gamma[x_t(z^{t-1}), z_t] \qquad , x_0, z_0 \text{ given}$$

- The Bellman Equation

$$v(x, z) = \max_{y \in \Gamma(x,z)} F(x, y, z) + \beta \sum_{z' \in Z} v(y, z') \pi(z' | z)$$

# 7.3. Stochastic Dynamic Programming

## Existence and uniqueness of the solution to Bellman Equations

- For existence and uniqueness, we need the same assumptions as before!
- The fact that $z$ is discrete is critical here. If $z$ is not discrete, we need to make sure that
  - The expectation operator maps continuous functions into continuous functions
  - $\Gamma(x, z)$ is continuous in $z$

- For (strict) monotonicity **in** $x$ we need the same assumptions as before:
  - $F$ is (strictly) increasing in $x$ for all $y, z$
  - $\Gamma$ is monotone in $x$ for all $z$
- (**NEW!**) For (strict) monotonicity **in** $z$ we need
  - $F$ is (strictly) increasing in $z$ for all $x, y$
  - $\Gamma$ is monotone in $z$ for all $x$ : $z \leq z' \Rightarrow \Gamma(x, z) \subseteq \Gamma(x, z')$ for all $x$
  - $\pi$ is monotone: If a function $f(z)$ is increasing in $z$ then a function

$$\hat{f}(z) = \sum_{z' \in Z} f(z') \pi(z'|z)$$

    is increasing in $z$.

- For (strict) concavity in $x$ and differentiability in $x$: the same assumptions as before:
- for (strict) concavity in $x$
  - $F$ is jointly (strictly) concave in $x, y$ **for all** $z$
  - $\Gamma$ is convex in $x$ **for all** $z$
- For differentiability in $x$:
  - all of the above
  - $F$ is differentiable in $x$ for all $y, z$
  - $g(x, z)$ is in the interior of $\Gamma(x, z)$

# Competitive Equilibria and Pareto Optima

## Introduction

| | | |
|---|---|---|
| **PO** | $\Longleftrightarrow$ | **RPO** |
| $\Updownarrow$ FWT / SWT | | $\Updownarrow$ |
| **CE** | $\Longleftrightarrow$ | **RCE** |

- Until now, we have studies Pareto Optima (PO) and Recursive Pareto Optima (RPO)
- We will now introduce the concept of **Competitive Equilibrium** (CE) and
  1. Study the connection with Pareto Optimum (welfare theorems)
  2. Find a recursive formulation of the competitive equilibrium (**Recursive Competitive Equilibrium**, RCE)

# Setup

- Stochastic production technology

$$y_t = z_t F(k_t, n_t)$$

  - $y_t$ is output, $k_t$ is capital, $n_t$ is labor
  - $z_t \in Z$ is productivity shock, Markov, discrete
  - $F$ is strictly increasing, strictly concave

- Household Preferences:

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U(c_t(z^t)) \Pi(z^t | z_0)$$

  - $c$ is consumption
  - $U$ is strictly increasing, strictly concave

- Initial capital stock and shock are given

# 1. Arrow-Debreu Equilibrium

- We will
  - Look at the trading arrangements
  - Look at the firm's problem
  - Look at the household's problem
  - Define the Arrow-Debreu Equilibrium
  - Look at its properties
    - Asset Pricing

# 1. Arrow-Debreu Equilibrium

## Prices

- All the trades are determined at time 0
- Future prices and trades are contingent on the history of shocks $z^t = (z_0, z_1, ....z_t)$
- **Complete markets**: for each $z^t$ there exists a market for consumption goods
- Prices:
  - $q_t(z^t)$ : Price of one unit of consumption in state $z^t$, in terms of time zero consumption
  - $w_t(z^t)$ : Price of one unit of labor in state $z^t$ in terms of state $z^t$ consumption
  - $r_t(z^t)$ : Rental price of one unit of capital in state $z^t$ in terms of state $z^t$ consumption

# 1. Arrow-Debreu Equilibrium

## Firm's Problem

- Firms make decision about
  - Capital demand $k_t^d(z^t)$
  - Labor demand $n_t^d(z^t)$
  - Output $y_t(z^t)$
- Firms maximize profits

$$
\begin{aligned}
P \quad = \quad & \max_{\{k^d, n^d, y\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[y_t(z^t) - r_t(z^t)k_t^d(z^t) - w_t(z^t)n_t^d(z^t)] \\
& s.t. \qquad y_t(z^t) \leq z_t F[k_t^d(z^t), n_t^d(z^t)]
\end{aligned}
$$

- Households decide about
  - Consumption $c_t(z^t)$
  - Capital supply $k_t^s(z^t)$
  - Labor supply $n_t^s(z^t)$
  - Investment $i_t(z^t)$
  - Capital holdings $x_{t+1}(z^t)$

given the initial capital holdings $x_0$.

# 1. Arrow-Debreu Equilibrium

## Household Problem

- Households maximize utility

$$\max_{\{k^s, n^s, c, i, x\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U(c_t(z^t)) \Pi(z^t | z_0)$$

s.t.

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[c_t(z^t) + i_t(z^t)] \leq \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[r_t(z^t) k_t^s(z^t) + w_t(z^t) n_t^s(z^t)] + P$$

$$x_{t+1}(z^t) = (1-\delta) x_t(z^{t-1}) + i_t(z^t)$$

$$0 \leq n_t^s(z^t) \leq 1$$

$$0 \leq k_t^s(z^t) \leq x_t(z^{t-1})$$

$$c_t(z^t) \geq 0$$

$$x_{t+1}(z^t) \geq 0$$

$$x_0 \text{ given}$$

## Definition

Arrow-Debreu Equilibrium is given by

1. a set of prices $(q, w, r)$
2. an allocation
   2.1 $(y, k^d, n^d)$ for the firm
   2.2 $(c, k^s, n^s, x)$ for the household

such that

1. $(y, k^d, n^d)$ solves the firm's problem given $(q, w, r)$
2. $(c, k^s, n^s, x)$ solves the household's problem given $(q, w, r)$
3. markets clear: for all $t, z^t$,

$$\begin{aligned} c_t(z^t) + i_t(z^t) &= y_t(z^t) && \text{(consumption goods mkt)} \\ n_t^s(z^t) &= n_t^d(z^t) && \text{(labor mkt)} \\ k_t^s(z^t) &= k_t^d(z^t) && \text{(capital mkt)} \end{aligned}$$

# 1. Arrow-Debreu Equilibrium

## Characterization

1a. First order conditions to the firm's problem: for all $t, z^t$,

$$
\begin{aligned}
z_t F_k[k_t^d(z^t), n_t^d(z^t)] &= r_t(z^t) > 0 \\
z_t F_n[k_t^d(z^t), n_t^d(z^t)] &= w_t(z^t) > 0
\end{aligned}
$$

- Factor prices are always strictly positive

1b. Since utility is strictly increasing, Arrow-Debreu prices are also always strictly positive: $q_t(z^t) > 0$ all $t, z^t$.

2. Assume that $F$ is CRS: $F(\lambda k, \lambda n) = \lambda F(k, n)$. Then $P = 0$.

3. Since factor prices are strictly positive (by 1a), we have

$$
\begin{aligned}
k_t^s(z^t) &= x_t(z^{t-1}) \\
n_t^s(z^t) &= 1
\end{aligned}
$$

The household problem becomes

$$
\max_{\{k^s, n^s, c, i, x\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U(c_t(z^t)) \Pi(z^t | z_0)
$$

s.t.

$$
\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[c_t(z^t) + i_t(z^t)] \leq \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} q_t(z^t)[(r_t(z^t) + 1 - \delta) k_t^s(z^{t-1}) + w_t(z^t)]
$$

$$
c_t(z^t) \geq 0, \; x_{t+1}(z^t) \geq 0, \; k_0 \text{ given}
$$

# 1. Arrow-Debreu Equilibrium

## Characterization

4. First order conditions for the household

$$\begin{aligned} \beta^t U'(c_t(z^t))\Pi(z^t|z_0) &= \lambda q_t(z^t) \\ \lambda[(r_t(z^t)+1-\delta)q_{t+1}(z^{t+1}) - q_t(z^t)] &\leq 0 \\ &= \text{ if } k_{t+1}(z^t) > 0 \end{aligned}$$

Pricing kernel:

$$q_t(z^t) = \beta^t \frac{U'(c_t(z^t))}{U'(c_0(z_0))}\Pi(z^t|z_0)$$

Price of an asset that delivers 1 unit of consumption in state $z^t$.

# 1. Arrow-Debreu Equilibrium

## Characterization: Asset pricing

- For each state $z^t$ there is one Arrow-Debreu asset, with price determined by

$$q_t(z^t) = \beta^t \frac{U'(c_t(z^t))}{U'(c_0(z_0))} \Pi(z^t|z_0)$$

- Advantage of the A-D trading mechanism: One can use "no arbitrage" argument to price any other asset

### Example

(risk free bond) An asset that delivers 1 unit of consumption in period 1 regardless of the state:

$$q_1^{RF} = \sum_{z_1 \in Z} q_1(z_0, z_1)$$

### Example

(riskless console) An asset that pays 1 unit of consumption forever