Dynamic Programming Under Certainty 6

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Today

- We will
  1. Look at the dynamics of the solution in the optimal growth model
  2. Look at stochastic dynamic programming

6.1. Example: Optimal Growth

$$v(k) = \max_{0 \leq y \leq f(k))} U(f(k) - y) + \beta v(y)$$

1. For the existence and uniqueness of the fixed point we need
   1.1 $\beta \in (0, 1)$
   1.2 For boundedness: Either U is bounded or there is an upper bound on capital stock $\bar{k}$ so that $k \in [0, \bar{k}]$. This is OK if $\lim_{k\to\infty} f'(k) = 0$.
   1.3 For TOM: U and f are continuous
2. For (strict) monotonicity of v we need
   2.1 U, f are (strictly) increasing
3. For strict concavity of v we need
   3.1 U, f are strictly concave
4. For differentiability of v we need
   4.1 U, f are differentiable and strictly concave
   4.2 For interior solution: $\lim_{c\to 0} U'(c) = +\infty$ and $\lim_{k\to 0} f'(k) = +\infty$.

6.1. Example: Optimal Growth

Optimal Policy Function

- $g(k)$ satisfies:
  - First order condition:

$$U'(f(k) - g(k)) = \beta v'(g(k))$$

  - Envelope condition:

$$v'(k) = U'(f(k) - g(k))f'(k)$$

- $g(k)$ is strictly increasing.

7.1. Dynamics in the Optimal Growth Model

- Is there a steady state?
- Is it unique?
- Is it stable? (Does the capital stock converge to the steady state?)

7.1. Dynamics in the Optimal Growth Model

Existence and Uniqueness of Steady State

- In steady state, $g(k^{ss}) = k^{ss}$. Hence

$$
\begin{aligned}
U'(f(k^{ss}) - k^{ss}) &= \beta v'(k^{ss}) \\
v'(k^{ss}) &= U'(f(k^{ss}) - k^{ss}) f'(k^{ss})
\end{aligned}
$$

- Hence

$$f'(k^{ss}) = \frac{1}{\beta}.$$

  - Steady state exists and is unique for a strictly positive capital stock.

7.1. Dynamics in the Optimal Growth Model

Stability of Steady State

- Since v is strictly concave,

$$[v'(k) - v'(\hat{k})][k - \hat{k}] \leq 0 \qquad \text{all } k, \hat{k}$$

  with equality only if $k = \hat{k}$.
- Choose $\hat{k} = g(k)$. Then

$$[v'(k) - v'(g(k))][k - g(k)] \leq 0$$

  with equality only in steady state.

7.1. Dynamics in the Optimal Growth Model

Stability of Steady State

- Using FOC and EC,

$$
\begin{aligned}
[U'(c)f'(k) - U'(c)\frac{1}{\beta}][k - g(k)] &\leq 0 \\
[f'(k) - \frac{1}{\beta}][k - g(k)] &\leq 0
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
g(k) &> k \Leftrightarrow f'(k) > \frac{1}{\beta} \Leftrightarrow k < k^{ss} \\
g(k) &< k \Leftrightarrow f'(k) < \frac{1}{\beta} \Leftrightarrow k > k^{ss}
\end{aligned}
$$

Optimal capital stock is decreasing if $k > k^{ss}$ and increasing if $k < k^{ss}$. Hence the solution is stable.

7.1. Stochastic Dynamic Programming

Example

Stochastic Optimal Growth:

$$y_t = \tilde{z}_t f(k_t)$$

$\{\tilde{z}_t\}_{t=0}^{\infty}$ is a sequence of (discrete) random variables

- $\tilde{z}_t$ is a **random variable** for all t.
  - Discrete: Can take values $\{z_1, z_2, ...z_I\} = Z$ for some $I \leq \infty$
- A sequence of random variables $\{\tilde{z}_t\}_{t=0}^{\infty}$ is called a **stochastic process**

7.1. Stochastic Dynamic Programming

- Two important cases:
  1. The stochastic process is **i.i.d.**

$$\Pr(\tilde{z}_t = z_t | \tilde{z}_{t-1} = z_{t-1}, ..\tilde{z}_0 = z_0) = \Pr(\tilde{z}_t = z_t)$$

  2. The stochastic process is **Markov**

$$\Pr(\tilde{z}_t = z_t | \tilde{z}_{t-1} = z_{t-1}, ..\tilde{z}_0 = z_0) = \Pr(\tilde{z}_t = z_t | \tilde{z}_{t-1} = z_{t-1})$$

7.1. Stochastic Dynamic Programming

Markov Shocks

- Define, for any $z^t = \{z_j\}_{j=0}^t$

$$\begin{aligned} \pi(z_t|z_{t-1}) &= \Pr(\tilde{z}_t = z_t|\tilde{z}_{t-1} = z_{t-1}) \\ \Pi(z^t|z_0) &= \Pr(\tilde{z}^t = z^t|\tilde{z}_0 = z_0) \end{aligned}$$

- Note that since the shocks are Markov,

$$\begin{aligned} \Pi(z^t|z_0) &= \Pr(\tilde{z}^t = z^t|\tilde{z}_0 = z_0) \\ &= \Pr(\tilde{z}^t = z^t|\tilde{z}^{t-1} = z^{t-1}) \Pr(\tilde{z}^{t-1} = z^{t-1}|\tilde{z}_0 = z_0) \\ &= \Pr(\tilde{z}_t = z_t|\tilde{z}^{t-1} = z^{t-1}) \Pr(\tilde{z}^{t-1} = z^{t-1}|\tilde{z}_0 = z_0) \\ &= \pi(z_t|z_{t-1})\Pi(z^{t-1}|z_0) \end{aligned}$$

- Therefore

$$\Pi(z^t|z_0) = \pi(z_t|z_{t-1})\pi(z_{t-1}|z_{t-2})..\pi(z_1|z_0)$$

7.2. Stochastic Dynamic Programming

Optimal Growth Problem

- The Sequence Problem

$$v^*(k_0, z_0) \quad = \quad \max_{\{k_{t+1}(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U[z_t f(k_t(z^{t-1})) - k_{t+1}(z^t)] \Pi(z^t|$$

$$s.t.\ 0 \quad \leq \quad k_{t+1}(z^t) \leq z_t f((z^{t-1})) \qquad k_0, z_0 \text{ given}$$

-
    - The optimal capital stock $k_{t+1}(z^t)$ and consumption $c_t(z^t)$ are now indexed by history of shocks z^t
- The Bellman Equation

$$v(k, z) = \max_{0 \leq y \leq zf(k)} U[zf(k) - y] + \beta \sum_{z' \in Z} v(y, z') \pi(z'|z)$$

  - The optimal policy function $g(k, z)$ now depends on the current shock z.

- We will show that the value function in the sequence problem $v^*(k, z)$ satisfies the Bellman Equation
- One can also show that under a certain boundedness condition, the solution to the Bellman Equation $v(k, z)$ satisfies the sequence problem.

7.2. Stochastic Dynamic Programming

A solution to (SP) satisfies (FE)

$$v^*(k_0, z_0)$$

$$= \max_{\{0 \le k_{t+1}(z^t) \le z_t f((z^{t-1}))\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t U[c_t(z^t)]\Pi(z^t|z_0)$$

$$= \max_{\{0 \le k_{t+1}(z^t) \le z_t f k_t((z^{t-1}))\}_{t=0}^{\infty}} \{U[z_0 f(k_0) - k_1(z_0)]$$

$$+\beta \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^{t-1} U[c_t(z^t)]\Pi(z^t|z_0)\}$$

$$= \max_{\{0 \le k_1(z) \le z_0 f(k_0)\}_{t=0}^{\infty}} \{U[z_0 f(k_0) - k_1(z_0)]$$

$$+\beta \max_{\{0 \le k_{t+1}(z^t) \le z_t f((z^{t-1}))\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \sum_{z^t \in Z^t} \beta^{t-1} U[c_t(z^t)]\Pi(z^t|z_0)\}$$

7.2. Stochastic Dynamic Programming

A solution to (SP) satisfies (FE)

Since $\Pi(z^t|z_0) = \Pi(z^t|z_1)\pi(z_1|z_0)$,

$$\begin{aligned}
& v^*(k_0, z_0) \\
= & \max_{\{0 \le k_1(z) \le z_0 f(k_0)\}_{t=0}^{\infty}} \{U[z_0 f(k_0) - k_1(z_0)] \\
& + \beta \sum_{z_1 \in Z} \{\max_{\{\cdot\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \sum_{z^t | z_1 \in Z^{t-1}} \beta^{t-1} U[c_t(z^t)] \Pi(z^t|z_1)\} \pi(z_1|z_0)\} \\
= & \max_{\{0 \le k_1(z) \le z_0 f(k_0)\}_{t=0}^{\infty}} \{U[z_0 f(k_0) - k_1(z_0)] + \beta \sum_{z_1 \in Z} v^*(k_1, z_1) \pi(z_1|z_0)
\end{aligned}$$

7.3. Stochastic Dynamic Programming

General Setup

- z can affect
  - the correspondence Γ
  - the objective function F
- The Sequence Problem

$$
\begin{aligned}
v^*(x_0, z_0) \quad &= \quad \max_{\{k_{t+1}(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t F[x_t(z^{t-1}), x_{t+1}(z^t), z_t] \Pi(z^t | z_0 \\
s.t. \; x_{t+1}(z^t) \quad &\in \quad \Gamma[x_t(z^{t-1}), z_t] \qquad x_0, z_0 \text{ given}
\end{aligned}
$$

- The Bellman Equation

$$
v(x, z) = \max_{y \in \Gamma(x,z)} F(x, y, z) + \beta \sum_{z' \in Z} v(y, z') \pi(z' | z)
$$

7.3. Stochastic Dynamic Programming

Existence and uniqueness of the solution to Bellman Equations

- For existence and uniqueness, we need the same assumptions as before!
- The fact that z is discrete is critical here. If z is not discrete, we need to make sure that
  - The expectation operator maps continuous functions into continuous functions
  - $\Gamma(x, z)$ is continuous in z

7.3. Stochastic Dynamic Programming

Properties of the value function

- For (strict) monotonicity **in** x we need the same assumptions as before:
  - F is (strictly) increasing in x for all y, z
  - Γ is monotone in x for all z
- (**NEW!**) For (strict) monotonicity **in** z we need
  - F is (strictly) increasing in z for all x, y
  - Γ is monotone in z for all x : $z \leq z' \Rightarrow \Gamma(x, z) \subseteq \Gamma(x, z')$ for all x
  - π is monotone: If a function $f(z)$ is increasing in z then a function
    $$\hat{f}(z) = \sum_{z' \in Z} f(z')\pi(z'|z)$$
    is increasing in z.

- For (strict) concavity in x and differentiability in x: the same assumptions as before:
- for (strict) concavity in x
  - F is jointly (strictly) concave in x, y **for all** z
  - Γ is convex in x **for all** z
- For differentiability in x:
  - all of the above
  - F is differentiable in x for all y, z
  - $g(x, z)$ is in the interior of $\Gamma(x, z)$