Dynamic Programming Under Certainty 5

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Review

- The Bellman Operator

$$(Tv)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$$

1. Has a **unique** fixed point v^* in the space of bounded and continuous functions
2. Sequence of functions $\{T^n v_0\}$ converges to v^* at rate β

- The optimal policy correspondence

$$g(x) = \arg\max_{y \in \Gamma(x)} F(x, y) + \beta v^*(y)$$

1. Is upper hemi-continuous and compact valued

Today

- We will
  1. Look at the properties of the fixed point
     a. Monotonicity
     b. Concavity
     c. Differentiability
  2. Look at the example of the optimal growth model again

5.1. Corollaries to the contraction mapping theorem

Corollary

Let (S, ρ) be a complete metric space. Let $T : S \to S$ be a contraction mapping that has a fixed point $v \in S$. Then

1. *If S' is a closed subset of S and $T(S') \subseteq S'$ then $v \in S'$.*
2. *If in addition $T(S') \subseteq S''$ then $v \in S''$.*

5.2 General Approach

- To show that the fixed point has a given property, we will
  - look at the conditions that guarantee that T maps a set of functions with that property onto itself
  - If the set of functions with a given property is closed, then, by Corollary 1, the fixed point will preserve that property
  - If the set of functions with a given property is not closed, but Corollary 2 holds, then fixed point will preserve the property.
- For differentiability, the approach fails: Corollaries 1 and 2 do not apply.
  - Other tricks

5.3. Monotonicity

- **Assumption M1:** $F(x, y)$ is increasing in x
  Assumption M2: Γ is monotone: $x \leq x' \Rightarrow \Gamma(x) \subseteq \Gamma(x')$

Theorem

If assumptions 1,2 and M hold then the fixed point v^* is increasing.

- Idea of proof: Let S^+ be a set of bounded, continuous and increasing functions. $S^+ \subseteq S$ and one can show that S^+ is closed. Under assumptions M T maps increasing, bounded and continuous functions onto itself. Corollary 1 can be applied.

5.3. Strict Monotonicity

Theorem

If in addition $F(x, y)$ is strictly increasing in x then the fixed point v^* is strictly increasing.

- Idea of proof: Let S^{++} be a set of bounded, continuous and strictly increasing functions. $S^{+} \subseteq S$. But since S^{++} is not closed, one must show that for any v increasing, Tv is strictly increasing. This is guaranteed by the assumption that F is strictly increasing.

5.4. (Strict) Concavity

- **Assumption C1:** $F(x, y)$ is jointly (strictly) concave in (x, y) :

$$F(\lambda(x, y) + (1 - \lambda)(x', y')) \geq \lambda F(x, y) + (1 - \lambda)F(x', y')$$

Assumption C2: Γ is convex: If $y \in \Gamma(x)$ and $y' \in \Gamma(x')$ then

$$\lambda y + (1 - \lambda)y' \in \Gamma(\lambda x + (1 - \lambda)x')$$

Theorem

If assumptions 1,2 and C hold then

1. *The fixed point v^* is (strictly) concave*
2. *If C1 is strict then the optimal policy function g is single valued and continuous*

5.5. Differentiability

- The usual approach relies on envelope theorems:

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

  - If f is differentiable .... then $h'(x) = f_x(x, g(x))$
  - one cannot apply directly since $f(x, y) \sim F(x, y) + \beta v(y)$ and we do not know in advance if $v(y)$ is differentiable.
- One cannot use the same approach as with monotonicity and concavity either
  - The space of bounded, continuous and differentiable functions is not closed (and Corollary 2 does not apply)

5.5. Differentiability

Benveniste-Schienkman

Theorem (Benveniste-Schienkman)

If there is a function $w(x)$ such that for some x_0

1. $w(x_0) = v(x_0)$ *and* $w(x) \leq v(x)$ *for* x *in the neighborhood of* x_0
2. v *and* w *are both concave*
3. w *is differentiable*

Then

1. v *is differentiable at* x_0,
2. $v'(x_0) = w'(x_0)$.

5.5. Differentiability

- candidate for w :

$$w(x) = F(x, g(x_0)) + \beta v(g(x_0))$$

  - policy function is fixed for x_0, x varies
- Check the conditions of B-S:

  1a. $w(x_0) = v(x_0)$.
  1b. **If** $g(x_0)$ is in the **interior of** $\Gamma(x_0)$ then $g(x_0) \in \Gamma(x)$ for x in the neighborhood of x_0 and so $w(x) \leq v(x)$.
  2. If Assumptions C hold (F is concave and Γ is convex) then both v and w are concave.
  3. If F is differentiable in x at x_0 then w is differentiable at x_0.
- New requirements: F must be differentiable in x and $g(x_0)$ must be in the interior of $\Gamma(x_0)$.

5.5. Differentiability

Theorem

If assumptions 1,2 and C holds, F is differentiable in x at x_0 and in addition $g(x_0)$ is in the interior of $\Gamma(x_0)$ then

1. *v is differentiable at x_0*
2. $$v'(x_0) = F_x(x_0, g(x_0))$$

6.1. Example: Optimal Growth

$$v(k) = \max_{0 \leq y \leq f(k))} U(f(k) - y) + \beta v(y)$$

1. For the existence and uniqueness of the fixed point we need
   1.1 $\beta \in (0, 1)$
   1.2 For boundedness: Either U is bounded or there is an upper bound on capital stock $\bar{k}$ so that $k \in [0, \bar{k}]$. This is OK if $\lim_{k \to \infty} f'(k) = 0$.
   1.3 For TOM: U and f are continuous
2. For (strict) monotonicity of v we need
   2.1 U, f are (strictly) increasing
3. For strict concavity of v we need
   3.1 U, f are strictly concave
4. For differentiability of v we need
   4.1 U, f are differentiable and strictly concave
   4.2 For interior solution: $\lim_{c \to 0} U'(c) = +\infty$ and $\lim_{k \to 0} f'(k) = +\infty$.

6.1. Example: Optimal Growth

Optimal Policy Function

- $g(k)$ satisfies:
  - First order condition:

$$U'(f(k) - g(k)) = \beta v'(g(k))$$

  - Envelope condition:

$$v'(k) = U'(f(k) - g(k))f'(k)$$

- $g(k)$ is strictly increasing.

7.1. Dynamics in the Optimal Growth Model

- Is there a steady state?
- Is it unique?
- Is it stable? (Does the capital stock converge to the steady state?)

7.1. Dynamics in the Optimal Growth Model

Existence and Uniqueness of Steady State

- In steady state, $g(k^{ss}) = k^{ss}$. Hence

$$\begin{aligned} U'(f(k^{ss}) - k^{ss}) &= \beta v'(k^{ss}) \\ v'(k^{ss}) &= U'(f(k^{ss}) - k^{ss}) f'(k^{ss}) \end{aligned}$$

- Hence

$$f'(k^{ss}) = \frac{1}{\beta}.$$

  - Steady state exists and is unique for a strictly positive capital stock.

7.1. Dynamics in the Optimal Growth Model

Stability of Steady State

- Since v is strictly concave,

$$[v'(k) - v'(\hat{k})][k - \hat{k}] \leq 0 \qquad \text{all } k, \hat{k}$$

  with equality only if $k = \hat{k}$.
- Choose $\hat{k} = g(k)$. Then

$$[v'(k) - v'(g(k))][k - g(k)] \leq 0$$

  with equality only in steady state.

7.1. Dynamics in the Optimal Growth Model

Stability of Steady State

- Using FOC and EC,

$$
\begin{aligned}
[U'(c)f'(k) - U'(c)\frac{1}{\beta}][k - g(k)] &\leq 0 \\
[f'(k) - \frac{1}{\beta}][k - g(k)] &\leq 0
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
g(k) &> k \Leftrightarrow f'(k) > \frac{1}{\beta} \Leftrightarrow k < k^{ss} \\
g(k) &< k \Leftrightarrow f'(k) < \frac{1}{\beta} \Leftrightarrow k > k^{ss}
\end{aligned}
$$

Optimal capital stock is decreasing if $k > k^{ss}$ and increasing if $k < k^{ss}$. Hence the solution is stable.