Dynamic Programming Under Certainty 4

Marek Kapicka, Econ 204b

April 8, 2009

Review

- Bellman Equation v^* as a fixed point

$$v^* = Tv^*$$

of the operator

$$(Tv)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$$

- Contraction Mapping Theorem: If T maps bounded and continuous functions to itself and is a contraction, it has a unique fixed point

Today

- We will
  1. Look at Sufficient conditions for a contraction
  2. Look at the Theorem of Maximum
  3. Apply CMT and TOM to the Bellman Operator
  4. Have Example
  5. Look at the properties of the fixed point
     5.1 Monotonicity

Theorem

Let S be a space of bounded functions on X, endowed with a sup norm. Let $T : S \to S$. If

*i) T is **monotone**: If $f(x) \leq g(x)$ for all $x \in X$ then $Tf(x) \leq Tg(x)$ for all $x \in X$.*

*ii) T **discounts**: For some $\beta \in (0, 1)$ and any $a \in R_+$,*

$$T(f + a)(x) \leq Tf(x) + \beta a \quad \forall x \in X,$$

where $(f + a)(x) = f(x) + a$,

then T is a contraction with modulus β.

4.3. Theorem of the Maximum

- We want to make sure that an operator T maps continuous functions into continuous functions.

- **Assumptions:**

  2a. Γ is nonempty (i.e. $\Gamma(x)$ is nonempty for all $x \in X$)
  2b. Γ is compact valued (i.e. $\Gamma(x)$ is compact for all $x \in X$)
  2c. Γ is continuous (??)

- Two weaker concepts:
  1. **upper hemi-continuity:** "no dips"
  2. **lower hemi-continuity**: "no spikes"

Definition

A correspondence is continuous if it is both u.h.c. and u.l.c.

4.3. Theorem of the Maximum

Theorem

Let $X \in R^l$ and $Y \in R^m$. Define

$$h(x) = \max_{y \in \Gamma(x)} f(x, y), \qquad g(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$$

Suppose that $f : X \times Y \rightarrow R$ is continuous and $\Gamma : X \rightarrow Y$ is nonempty, compact valued and continuous. Then
i) $h : X \rightarrow R$ is continuous and
ii) $g : X \rightarrow Y$ is upper hemi-continuous and compact valued.

4.4. Bellman Equation Application

- The Bellman Operator:

$$(Tv)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$$

Theorem

Let S be the space of bounded and continuous functions with a sup norm. Suppose that:
i) (A1): $F(x, y)$ is bounded and continuous
ii) $0 < \beta < 1$
iii) (A2): Γ is nonempty, compact valued and continuous.
Then the Bellman operator T
i) maps S onto itself,
ii) has a unique fixed point $v^* \in S$,
iii) $\| T^n v_0 - v^* \| \leq \beta^n \| v_0 - v^* \|$,
iv) The optimal policy correspondence $g(x)$ is compact valued and u.h.c.

4.4. Bellman Equation Application

Proof.
1. By TOM T maps continuous functions into continuous functions
2. T is a contraction: (Blackwell):
(monotonicity): obvious
(discounting):

$$\begin{aligned} T(v+a)(x) &= \max_{y\in\Gamma(x)}[F(x,y)+\beta(v+a)(y)] \\ &= \max_{y\in\Gamma(x)}[F(x,y)+\beta v(y)+\beta a] \\ &= \max_{y\in\Gamma(x)}[F(x,y)+\beta v(y)]+\beta a \\ &= Tv(x)+\beta a \end{aligned}$$

□

5.1. Corollaries to the contraction mapping theorem

Corollary

Let (S, ρ) be a complete metric space. Let $T : S \to S$ be a contraction mapping that has a fixed point $v \in S$. Then

1. *If S' is a closed subset of S and $T(S') \subseteq S'$ then $v \in S'$.*
2. *If in addition $T(S') \subseteq S''$ then $v \in S''$.*

5.2 General Approach

- to show that the fixed point has a given property, we will
  - look at the conditions that guarantee that T maps a set of functions with that property onto itself
  - If the set of functions with a given property is closed, then, by Corollary 1, the fixed point will preserve that property
  - If the set of functions with a given property is not closed, but Corollary 2 holds, then fixed point will preserve the property.
- For differentiability, the approach fails: Corollaries 1 and 2 do not apply.
  - Other tricks

5.3. Monotonicity

- **Assumption M1:** $F(x, y)$ is increasing in x
  Assumption M2: Γ is monotone: $x \leq x' \Rightarrow \Gamma(x) \subseteq \Gamma(x')$

Theorem

If assumptions 1,2 and M hold then the fixed point v^* is increasing.

- Idea of proof: Let S^+ be a set of bounded, continuous and increasing functions. $S^+ \subseteq S$ and one can show that S^+ is closed. Under assumptions M T maps increasing, bounded and continuous functions onto itself. Corollary 1 can be applied.

5.3. Strict Monotonicity

Theorem

If in addition $F(x, y)$ is strictly increasing in x then the fixed point v^* is strictly increasing.

- Idea of proof: Let S^{++} be a set of bounded, continuous and strictly increasing functions. $S^+ \subseteq S$. But since S^{++} is not closed, one must show that for any v increasing, Tv is strictly increasing. This is guaranteed by the assumption that F is strictly increasing.