Dynamic Programming Under Certainty Contd

Marek Kapicka, Econ 204b

April 6, 2009

Review

- Bellman Equation v^* as a fixed point

$$v^* = Tv^*$$

  of the operator

$$(Tv)(k) = \max_{0 \leq y \leq f(k)} U(f(k) - y) + \beta v(y)$$

- **Today**

  3.1 Finish the Example
  4. General Approach

3.1. An Example

- $u(c) = \ln c,\ f(k) = k^{\alpha}$ (full depreciation)

1. **STEP 1**
   Set $v_0 = 0$:
   $$Tv_0(k) = \max_{0 \le y \le k^{\alpha}} \ln(k^{\alpha} - y)$$
   Solution:
   $$\begin{aligned} g_0(k) &= 0 \\ Tv_0(k) &= \ln k^{\alpha} = \alpha \ln k \end{aligned}$$

3.1. An Example

2. **STEP 2**

   Set $v_1 = Tv_0 = \alpha \ln k$:

$$Tv_1(k) = \max_{0 \leq y \leq k^\alpha} \ln(k^\alpha - y) + \alpha\beta \ln y$$

   Solution:

$$\begin{aligned} g_1(k) &= \frac{\alpha\beta}{1+\alpha\beta} k^\alpha \\ Tv_1(k) &= \alpha(1+\alpha\beta) \ln k + \ln \frac{1}{1+\alpha\beta} + \alpha\beta \ln \frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

3.1. An Example

3. **STEP 3**

Set $v_2 = Tv_1 = \alpha(1+\alpha\beta)\ln k + \ln\frac{1}{1+\alpha\beta} + \alpha\beta\ln\frac{\alpha\beta}{1+\alpha\beta}$:

$$\begin{aligned} Tv_2(k) &= \max_{0\le y\le k^\alpha} \ln(k^\alpha - y) + \alpha\beta(1+\alpha\beta)\ln y \\ &\quad + \ln\frac{\beta}{1+\alpha\beta} + \alpha\beta^2\ln\frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

Solution:

$$\begin{aligned} g_2(k) &= \frac{\alpha\beta + (\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}k^\alpha \\ Tv_2(k) &= \alpha(1+\alpha\beta+(\alpha\beta)^2)\ln k \\ &\quad + \ln\frac{1}{1+\alpha\beta+(\alpha\beta)^2} + (\alpha\beta+(\alpha\beta)^2)\ln\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} \\ &\quad + \beta\ln\frac{1}{1+\alpha\beta} + \alpha\beta^2\ln\frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

3.1. An Example

∞ :**THE LIMIT**

$$v^* = \lim_{s\to\infty} v_s$$
$$g^* = \lim_{s\to\infty} g_s$$

$$v^*(k) = \frac{\alpha}{1-\beta}\ln k + \frac{1}{1-\beta}[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta]$$
$$g^*(k) = \alpha\beta k^\alpha$$

Hence, $c^*(k) = k^\alpha - g^*(k) = (1-\alpha\beta)k^\alpha$

- Consume a fraction $1-\alpha\beta$ of the resources, save $\alpha\beta$ for the future.

3.1. An Example

Results

- By iteration on the value function, we have found
  1. The value function v^* that solves (FE)
  2. The optimal policy function $g^*(k)$
- Note: The speed of convergence

$$
\begin{aligned}
&\alpha \ln k \\
&\alpha(1+\alpha\beta) \ln k \\
&\alpha(1+\alpha\beta+(\alpha\beta)^2) \ln k \\
&\dots
\end{aligned}
$$

will hold more generally!

4. A General Approach

- Consider the following more general setup: $x \in X$,

$$
\begin{aligned}
(SP): v^*(x) &= \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
& s.t. \quad x_{t+1} \in \Gamma(x_t) \\
& x_0 \text{ given}
\end{aligned}
$$

$$
(FE): v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)
$$

- Γ is a correspondence: Assigns a set $\Gamma(x)$ to each x.

4.1. A General Approach - Mathematical Preliminaries

- **Assumption:** $F(x, y)$ is bounded and continuous.
- The assumption suggests that the value function might also be bounded and continuous
- To study (FE) in general, we need some mathematical tools:
  - *Contraction Mapping Theorem*: Will tell us under what conditions will an operator mapping the set of bounded and continuous functions onto itself have a unique fixed point.
  - *Theorem of Maximum*: Will tell us under what conditions will the Bellman operator map the set of bounded and continuous functions onto itself.

4.1. A General Approach - Mathematical Preliminaries

- Consider a set of bounded and continuous functions with a *sup norm*

$$S = \{f : X \to R,\ f \text{ is continuous}$$
$$\| f \| = \sup_{x \in X} |f(x)| < \infty\}$$

- Define a metric

$$\rho(f, g) = \| f - g \| = \sup_{x \in X} |f(x) - g(x)|$$

Definition

A metric space (S, ρ) is **complete** if every Cauchy sequence in S converges to an element in S.

4.1. A General Approach - Mathematical Preliminaries

Definition

A sequence $\{x_n\}$ is a **Cauchy sequence** if for all $\varepsilon > 0$ $\exists N_\varepsilon$ such that $\rho(x_m, x_n) < \varepsilon$ for all $m, n \geq N_\varepsilon$.

Theorem

The set of bounded and continuous functions is complete

4.1. Example: A set of functions that is not complete

- Let S^{++} be a set of bounded continuous functions that are *strictly* increasing
- Consider a sequence

$$f_n(x) = 1 + \frac{1}{1+n}x \qquad x \in [0,1]$$

- $\{f_n\}$ is a sequence of functions

4.1. Example: A set of functions that is not complete

- $\{f_n\}$ is a Cauchy sequence: for any $\varepsilon > 0$,

$$\begin{aligned} \rho(f_m, f_n) &= \sup_{x\in[0,1]} |1 + \frac{1}{1+m}x - 1 - \frac{1}{1+n}x \\ &= |\frac{1}{1+m} - \frac{1}{1+n}| \\ &= \frac{1}{1+\min(m,n)} < \varepsilon \end{aligned}$$

  for $m, n \geq N_\varepsilon$ where $N_\varepsilon = \frac{1}{\varepsilon} - 1$.
- However,

$$\lim_{n\to\infty} f_n(x) = 1$$

  which is not strictly increasing. Hence S^{++} is not complete.

4.2. Contraction Mapping

Definition

Let (S, ρ) be a complete metric space. Let $T : S \to S$ be an operator. T is a **contraction with modulus** $\beta \in (0, 1)$ if

$$\rho(Tf, Tg) \leq \beta\rho(f, g) \qquad \text{any } f, g \in S$$

Example

$S = R$, $f(x) = \alpha + \beta x$, $\beta \in (0, 1)$.

$$\begin{aligned} \rho(f(x), f(y)) &= \rho(\alpha + \beta x, \alpha + \beta y) \\ &= \beta\rho(x, y). \end{aligned}$$

Example

S is a space of bounded, continuous functions, $Tf = \alpha + \beta f$.

Theorem

If T is a contraction in (S, ρ) with modulus β, then
i) There is a unique fixed point $f^* \in S$, $f^* = Tf^*$.
ii) Iterations of T converge to the fixed point:

$$\rho(T^n f_0, f^*) \leq \beta^n \rho(f_0, f^*)$$

for any $f_0 \in S$, where $T^{n+1} f_0 = T(T^n f_0)$.

Theorem

Let S be a space of bounded functions on X, endowed with a sup norm. Let $T : S \to S$. If

i) T is **monotone**: *If $f(x) \leq g(x)$ for all $x \in X$ then $Tf(x) \leq Tg(x)$ for all $x \in X$.*

ii) T **discounts**: *For some $\beta \in (0, 1)$ and any $a \in R_+$,*

$$T(f + a)(x) \leq Tf(x) + \beta a \quad \forall x \in X,$$

where $(f + a)(x) = f(x) + a$,

then T is a contraction with modulus β.

4.3. Theorem of the Maximum

- We want to make sure that an operator T maps continuous functions into continuous functions.

- **Assumptions:**

  2a. Γ is nonempty (i.e. $\Gamma(x)$ is nonempty for all $x \in X$)
  2b. Γ is compact valued (i.e. $\Gamma(x)$ is compact for all $x \in X$)
  2c. Γ is continuous (??)

- Two weaker concepts:
  1. **upper hemi-continuity:** ”no dips”
  2. **lower hemi-continuity**: ”no spikes”

Definition

A correspondence is continuous if it is both u.h.c. and u.l.c.

4.3. Theorem of the Maximum

Theorem

Let $X \in R^l$ and $Y \in R^m$. Define

$$h(x) = \max_{y \in \Gamma(x)} f(x, y), \qquad g(x) = \arg\max_{y \in \Gamma(x)} f(x, y)$$

Suppose that $f : X \times Y \to R$ is continuous and $\Gamma : X \to Y$ is nonempty, compact valued and continuous. Then
i) $h : X \to R$ is continuous and
ii) $g : X \to Y$ is upper hemi-continuous and compact valued.

4.4. Bellman Equation Application

- The Bellman Operator:

$$(Tv)(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)$$

Theorem

Let S be the space of bounded and continuous functions with a sup norm. Suppose that:
i) (A1): $F(x, y)$ is bounded and continuous
ii) $0 < \beta < 1$
iii) (A2): Γ is nonempty, compact valued and continuous.
Then the Bellman operator T
i) maps S onto itself,
ii) has a unique fixed point $v^* \in S$,
iii) $\| T^n v_0 - v^* \| \leq \beta^n \| v_0 - v^* \|$,
iv) The optimal policy correspondence $g(x)$ is compact valued and u.h.c.

4.4. Bellman Equation Application

Proof.
1. By TOM T maps continuous functions into continuous functions
2. T is a contraction: (Blackwell):
(monotonicity): obvious
(discounting):

$$\begin{aligned} T(v+a)(x) &= \max_{y \in \Gamma(x)} [F(x,y) + \beta(v+a)(y)] \\ &= \max_{y \in \Gamma(x)} [F(x,y) + \beta v(y) + \beta a] \\ &= \max_{y \in \Gamma(x)} [F(x,y) + \beta v(y)] + \beta a \\ &= Tv(x) + \beta a \end{aligned}$$

□