Dynamic Programming Under Certainty Contd

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April 1, 2009

Review

- Functional Equation and Sequence Problem:

$$(SP): v^*(k_0) = \max_{\{0 \le k_{t+1} \le f(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}),\ k_0 \text{ given}$$

$$(FE): v(k) = \max_{0 \le y \le f(k)} U(f(k) - y) + \beta v(y)$$

- **Today**

2. Connection between (SP) and (FE)
3. Bellman Equation as a Fixed Point

2.2a. A solution to (SP) satisfies (FE):

$$
\begin{aligned}
v^*(k_0) &= \max_{\{0 \le k_{t+1} \le f(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \\
&= \max_{\{0 \le k_{t+1} \le f(k_t)\}_{t=0}^{\infty}} \left[U(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1}) \right] \\
&= \max_{0 \le k_1 \le f(k_0)} \left[U(f(k_0) - k_1) \right. \\
&\quad \left. + \beta \max_{\{0 \le k_{t+1} \le f(k_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1}) \right] \\
&= \max_{0 \le k_1 \le f(k_0)} \left[U(f(k_0) - k_1) + \beta v^*(k_1) \right]
\end{aligned}
$$

Theorem

v^* satisfies (FE)

2.2b. A solution to (FE) satisfies (SP):

$$
\begin{aligned}
v(k_0) &= \max_{0 \le k_1 \le f(k_0)} [U(f(k_0) - k_1) + \beta v(k_1)] \\
&= \max_{0 \le k_1 \le f(k_0)} \{U(f(k_0) - k_1) \\
&\quad + \beta \max_{0 \le k_2 \le f(k_1)} [U(f(k_1) - k_2) + \beta v(k_2)]\} \\
&= \max_{\{0 \le k_{t+1} \le f(k_t)\}_{t=0}^{1}} \{\sum_{t=0}^{1} \beta^t [U(f(k_t) - k_{t+1}) + \beta^2 v(k_2)\} \\
&\quad \cdots\cdot \\
&= \max_{\{0 \le k_{t+1} \le f(k_t)\}_{t=0}^{T}} \{\sum_{t=0}^{T} \beta^t [U(f(k_t) - k_{t+1}) + \beta^{T+1} v(k_{T+1})\}
\end{aligned}
$$

2.2b. A solution to (FE) satisfies (SP): harder

Theorem

If

$$\lim_{T\to\infty} \beta^{T+1} v(k_{T+1}) = 0 \tag{1}$$

for all $\{k_{T+1}\}$ *such that* $0 \leq k_{T+1} \leq f(k_T)$, *then* v *satisfies (SP)*

Proof.

If the condition is satisfied then the last term vanishes:

$$\begin{aligned} v(k_0) &= \lim_{T\to\infty} \max_{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{T}} \{\sum_{t=0}^{T} \beta^t [U(f(k_t) - k_{t+1}) + \beta^{T+1} v(k_{T+1})\} \\ &= \max_{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty}} \{\sum_{t=0}^{\infty} \beta^t [U(f(k_t) - k_{t+1}) \end{aligned}$$

$\square$

2.1contd. The second Example revisited

- By Theorem 2, it must be true that the condition fails for some feasible sequence of capital
- Consider the following feasible sequence:

$$\begin{aligned} k_1 &= \frac{1}{\beta} k_0 \\ k_2 &= \frac{1}{\beta^2} k_0 \\ &\dots \end{aligned}$$

- Then

$$\begin{aligned} \lim_{T\to\infty} \beta^{T+1} v(k_{T+1}) &= \lim_{T\to\infty} \beta^{T+1} [\frac{1}{\beta^{T+1}} k_0] \\ &= k_0 \neq 0 \end{aligned}$$

- Hence Condition (1) is not satisfied for $v(k) = k$
- Note that Condition (1) is also not satisfied for the right solution!!
  - Condition (1) is sufficient but not necessary.

2.3. A Relationship between optimal policies in (SP) and (FE)

Theorem

If $\{k^*_{t+1}\}$ *attains the maximum of (SP) then* $k^*_{t+1} = g(k^*_t)$ *all* $t \geq 0$

Theorem

If

$$\lim_{T\to\infty} \beta^{T+1} v(k_{T+1}) \leq 0$$

then $\{g(k_0).g(g(k_0)), ...\}$ *attains the maximum of (SP).*

3. Bellman Equation as a Fixed Point

- Let S be a space of functions. Define a Bellman operator $T : S \to S$ by

$$Tv(k) = \max_{0 \le y \le f(k)} U(f(k) - y) + \beta v(y)$$

  - T maps a family of functions to a family of functions.
- The solution to (FE) v^* is a fixed point of the operator T :

$$Tv^* = v^*$$

  - If we find a fixed point of the operator T, we have solved (FE) and, provided the boundedness condition 2 is satisfied, (SP)

3. Bellman Equation as a Fixed Point

- We will show
  - (under certain conditions) there exists a **unique** fixed point
  - A sequence $\{v_{s+1}\}$ defined by

    $$v_{s+1} = Tv_s$$

    converges to the fixed point (useful computationally)
  - other properties of the fixed point (monotonicity, concavity, etc.)

3.1. An Example

- $u(c) = \ln c$, $f(k) = k^{\alpha}$ (full depreciation)

1. **STEP 1**

   Set $v_0 = 0$:

$$Tv_0(k) = \max_{0 \le y \le k^{\alpha}} \ln(k^{\alpha} - y)$$

   Solution:

$$\begin{aligned} g_0(k) &= 0 \\ Tv_0(k) &= \ln k^{\alpha} = \alpha \ln k \end{aligned}$$

3.1. An Example

2. **STEP 2**

   Set $v_1 = Tv_0 = \alpha \ln k$:

$$Tv_1(k) = \max_{0 \le y \le k^\alpha} \ln(k^\alpha - y) + \alpha\beta \ln y$$

   Solution:

$$\begin{aligned} g_1(k) &= \frac{\alpha\beta}{1+\alpha\beta} k^\alpha \\ Tv_1(k) &= \alpha(1+\alpha\beta)\ln k + \ln \frac{1}{1+\alpha\beta} + \alpha\beta \ln \frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

3.1. An Example

3. **STEP 3**

Set $v_2 = Tv_1 = \alpha(1+\alpha\beta)\ln k + \ln\frac{1}{1+\alpha\beta} + \alpha\beta\ln\frac{\alpha\beta}{1+\alpha\beta}$:

$$\begin{aligned} Tv_2(k) &= \max_{0\le y\le k^\alpha} \ln(k^\alpha - y) + \alpha\beta(1+\alpha\beta)\ln y \\ &\quad + \ln\frac{\beta}{1+\alpha\beta} + \alpha\beta^2\ln\frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

Solution:

$$\begin{aligned} g_2(k) &= \frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}k^\alpha \\ Tv_2(k) &= \alpha(1+\alpha\beta+(\alpha\beta)^2)\ln k \\ &\quad + \ln\frac{1}{1+\alpha\beta+(\alpha\beta)^2} + (\alpha\beta+(\alpha\beta)^2)\ln\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} \\ &\quad + \beta\ln\frac{1}{1+\alpha\beta} + \alpha\beta^2\ln\frac{\alpha\beta}{1+\alpha\beta} \end{aligned}$$

3.1. An Example

∞ :**THE LIMIT**

$$\begin{aligned} v^* &= \lim_{s\to\infty} v_s \\ g^* &= \lim_{s\to\infty} g_s \end{aligned}$$

$$\begin{aligned} v^*(k) &= \frac{\alpha}{1-\beta}\ln k + \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln\alpha\beta\right] \\ g^*(k) &= \alpha\beta k^\alpha \end{aligned}$$

Hence, $c^*(k) = k^\alpha - g^*(k) = (1-\alpha\beta)k^\alpha$

- Consume a fraction $1-\alpha\beta$ of the resources, save $\alpha\beta$ for the future.

3.1. An Example

Results

- By iteration on the value function, we have found
  1. The value function v^* that solves (FE)
  2. The optimal policy function $g^*(k)$
- Note: The speed of convergence

$$
\begin{aligned}
&\alpha \ln k \\
&\alpha(1+\alpha\beta) \ln k \\
&\alpha(1+\alpha\beta+(\alpha\beta)^2) \ln k \\
&\ldots
\end{aligned}
$$

will hold more generally!

4. A General Approach

- Consider the following more general setup: $x \in X$,

$$
\begin{aligned}
(SP) : v^*(x) \quad = \quad & \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
& s.t. \quad x_{t+1} \in \Gamma(x_t) \\
& x_0 \text{ given}
\end{aligned}
$$

$$
(FE) : v(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y)
$$

- Γ is a correspondence: Assigns a set $\Gamma(x)$ to each x.

4.1. A General Approach - Mathematical Preliminaries

- **Assumption:** $F(x, y)$ is bounded and continuous.
- The assumption suggests that the value function might also be bounded and continuous
- To study (FE) in general, we need some mathematical tools:
  - *Contraction Mapping Theorem*: Will tell us under what conditions will an operator mapping the set of bounded and continuous functions onto itself have a unique fixed point.
  - *Theorem of Maximum*: Will tell us under what conditions will the Bellman operator map the set of bounded and continuous functions onto itself.

4.1. A General Approach - Mathematical Preliminaries

- Consider a set of bounded and continuous functions with a *sup norm*

$$S = \{ \quad f : X \to R, \; f \text{ is continuous} \\ \| \; f \; \| = \sup_{x \in X} |f(x)| < \infty \}$$

- Define a metric

$$\rho(f, g) = \| \, f - g \, \| = \sup_{x \in X} |f(x) - g(x)|$$

Definition

A metric space (S, ρ) is **complete** if every Cauchy sequence in S converges to an element in S.

4.1. A General Approach - Mathematical Preliminaries

Definition

A sequence $\{x_n\}$ is a **Cauchy sequence** if for all $\varepsilon > 0$ $\exists N_\varepsilon$ such that $\rho(x_m, x_n) < \varepsilon$ for all $m, n \geq N_\varepsilon$.

Theorem

The set of bounded and continuous functions is complete

Example

A set of functions that is not complete

4.1. Example: A set of functions that is not complete

- Let S^{++} be a set of bounded continuous functions that are *strictly* increasing
- Consider a sequence

$$f_n(x) = 1 + \frac{1}{1+n}x \qquad x \in [0,1]$$

- $\{f_n\}$ is a sequence of functions
- $\{f_n\}$ is a Cauchy sequence: for any $\varepsilon > 0$,

$$\begin{aligned} \rho(f_m, f_n) &= \sup_{x\in[0,1]} |1 + \frac{1}{1+m}x - 1 - \frac{1}{1+n}x \\ &= |\frac{1}{1+m} - \frac{1}{1+n}| \\ &= \frac{1}{1+\min(m,n)} < \varepsilon \end{aligned}$$

for $m, n \geq N_\varepsilon$ where $N_\varepsilon = \frac{1}{\varepsilon} - 1$.

4.1. Example: A set of functions that is not complete

- However,
$$\lim_{n\to\infty} f_n(x) = 1$$
which is not strictly increasing. Hence S^{++} is not complete.