Real Business Cycles

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4. Linear-Quadratic Approximation

4. Linear Quadratic Approximation
    4.1. Form the linear-quadratic approximation of the period return function ✓
    4.2. Rearrange the terms ✓
    4.3. Guess that the approximated value function is quadratic in the states
    4.4. Derive the Riccatti equation that solves for the value function
5. The Results

4. Linear-Quadratic Approximation

4.3. Guess that the value function is quadratic

- The approximated objective function is quadratic in x, u :

$$\tilde{r}(y) = x^T Q_{xx} x + u^T Q_{uu} u + 2u^T Q_{xu} x$$

- The constraints are linear in x, u, ε :

$$x' = Ax + Bu + C\varepsilon$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \frac{1-\delta}{(1+\gamma)(1+\eta)} \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{(1+\gamma)(1+\eta)} \end{bmatrix}, C = \begin{bmatrix} 0 \\ \nu \\ 0 \end{bmatrix}$$

- Guess: the value function of the approximated problem is quadratic in x :

$$\tilde{V}(k, \varepsilon) = x^T P x$$

for some symmetric matrix P.

- If we find the matrix P, we solve the problem

4. Linear-Quadratic Approximation

4.3. Certainty equivalence

- The P matrix satisfies

$$x^T P x = \max_u \{x^T Q_{xx} x + u^T Q_{uu} u + 2u^T Q_{xu} x + \hat{\beta} E(x'^T P x' | x)\}$$

s.t.

$$x' = Ax + Bu + C\varepsilon.$$

where $\hat{\beta} = \beta(1+\eta)$

  - The solution is linear in the states: For some matrix F,

$$u = Fx.$$

- **Certainty Equivalence:** The solution is *independent* of the vector C
  - one can take $C = 0$ (no *perceived* uncertainty) and will get exactly the same solution

4. Linear-Quadratic Approximation

4.4. The Riccati Equation

- Using the certainty equivalence, it is enough to solve the following problem:

$$x^T P x = \max_u \{x^T Q_{xx} x + u^T Q_{uu} u + 2u^T Q_{xu} x + \hat{\beta} x'^T P x'\}$$

s.t.

$$x' = Ax + Bu.$$

- Taking first order conditions, one gets that

$$u = -(Q_{uu} + \hat{\beta} B^T P B)^{-1} (\hat{\beta} B^T P A + Q_{xu}) x.$$

- Substituting into the objective function we get that P solves the **Riccati equation**

$$\begin{aligned} P &= Q_{xx} + \hat{\beta} A^T P A \\ &\quad - (\hat{\beta} B^T P A + Q_{xu})^T (Q_{uu} + \hat{\beta} B^T P B)^{-1} (\hat{\beta} B^T P A + Q_{xu}) \end{aligned}$$

4. Linear-Quadratic Approximation

4.4. Numerical implementation

1. Start with some symmetric negative semi-definite matrix P_1
2. For a given matrix P_j, use the Riccati difference equation

$$\begin{aligned} P_{j+1} = & \; Q_{xx} + \hat{\beta} A^T P_j A \\ & - (\hat{\beta} B^T P_j A + Q_{xu})^T (Q_{uu} + \hat{\beta} B^T P_j B)^{-1} (\hat{\beta} B^T P_j A + Q_{xu}) \end{aligned}$$

   to obtain a new matrix P_{j+1}.
3. Iterate until $\| P_{j+1} - P_j \| \leq$ error tolerance
4. the iteration takes 0.31 second :-)

5. Real Business Cycle Model - How it works

- Denote the interest rate and the wage rate

$$R = \alpha e^{\omega}(\frac{n}{k})^{1-\alpha} + 1 - \delta$$
$$W = (1-\alpha)e^{\omega}(\frac{n}{k})^{-\alpha}$$

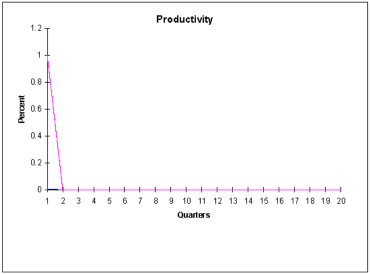
The first order conditions imply

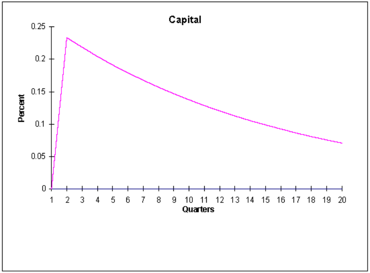
$$\beta E(\frac{c}{c'}R'|\omega) = 1 + \gamma$$
$$\beta E(\frac{1-n}{1-n'}\frac{W}{W'}R'|\omega) = 1 + \gamma$$

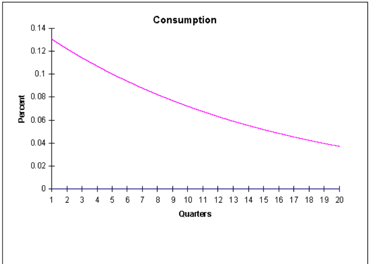
- Agents' current consumption is relatively low whenever the expected interest rate is high
- Agents' current labor supply is relatively high whenever the expected interest rate is high and/or current wage rate is relatively high
  - the importance of intertemporal effects on labor supply

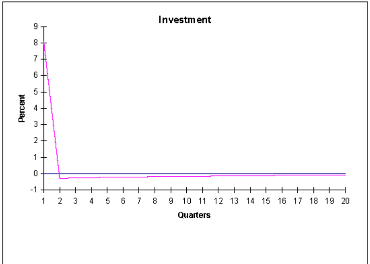
5. Real Business Cycle Model - How it works

Impulse Response to Temporary Shocks



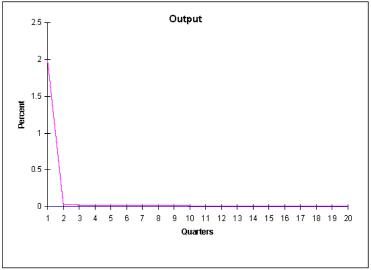


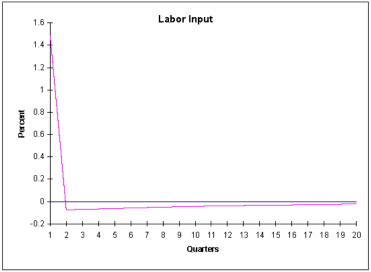


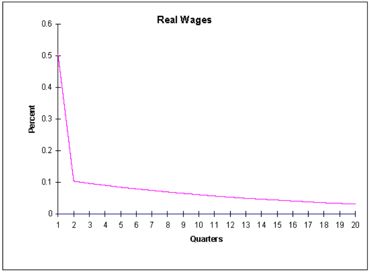


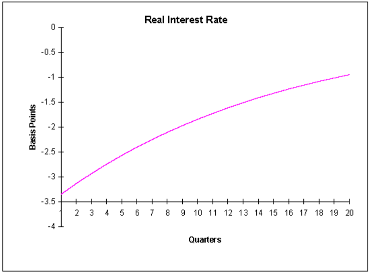
5. Real Business Cycle Model - How it works

Impulse Response to Temporary Shocks



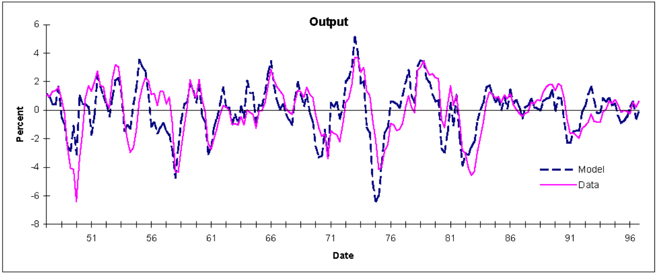


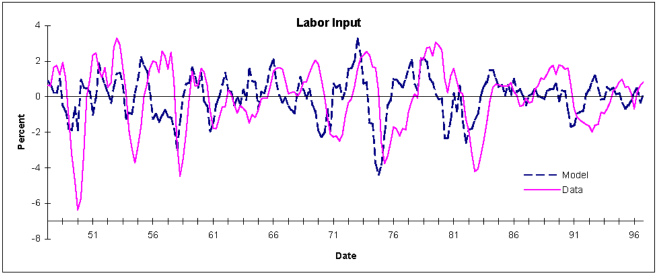




5. Real Business Cycle Model - Results

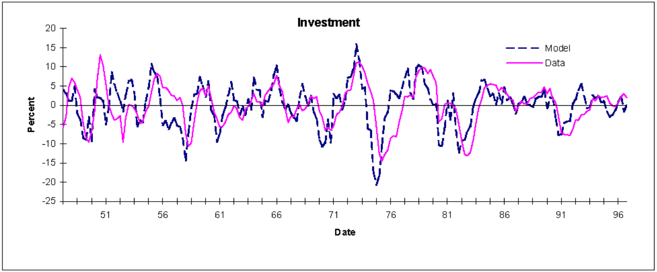
Time Series

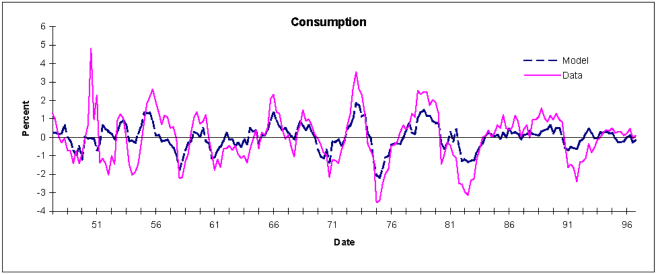




5. Real Business Cycle Model - Results

Time Series





5. Real Business Cycle Model - Results

US data

Table 1
Business Cycle Statistics for the U.S. Economy

	Standard Deviation	Relative Standard Deviation	First Order Auto-correlation	Contemporaneous Correlation with Output
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

Figure:

5. Real Business Cycle Model - Results

Model generated data

Table 3
Business Cycle Statistics for Basic RBC Model[35]

	Standard Deviation	Relative Standard Deviation	First Order Auto-correlation	Contemporaneous Correlation with Output
Y	1.39	1.00	0.72	1.00
C	0.61	0.44	0.79	0.94
I	4.09	2.95	0.71	0.99
N	0.67	0.48	0.71	0.97
Y/N	0.75	0.54	0.76	0.98
w	0.75	0.54	0.76	0.98
r	0.05	0.04	0.71	0.95
A	0.94	0.68	0.72	1.00

Figure:

5. Real Business Cycle Model - Results

- Overall, RBC model provides reasonable fit for the data
- It explains

$$\frac{1.39}{1.81} \approx 77\%$$

  of business cycles
- On several dimensions the model misses:
  1. the consumption is too smooth
  2. labor supply is too smooth
  3. interest rate is procyclical
  4. real wage is too volatile

5. Real Business Cycle Model - what drives the results

1. The productivity shocks must be **large**
2. The productivity shocks must be **persistent**
   - The model is very sensitive to changes in persistency of shocks
3. The Elasticity of Labor must be **large**

5. Real Business Cycle Model - what drives the results

Small Productivity Shocks

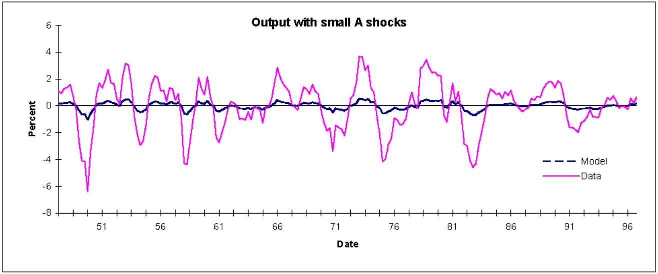


Figure:

5. Real Business Cycle Model - what drives the results

Small Elasticity of Labor

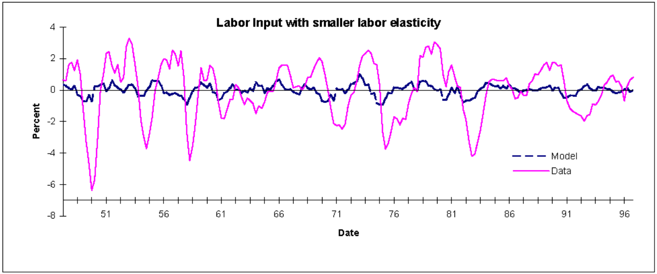


Figure: